A logical framework for incremental type-checking

Matthias Puech[1,2] Yann Régis-Gianas[2]

[1]Dept. of Computer Science, University of Bologna

[2]University Paris 7, CNRS, and INRIA, PPS, team πr^2

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PPS – Groupe de travail théorie des types et réalisabilité

A paradoxical situation

Observation

We have powerful tools to mechanize the metatheory of (proof) languages

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Workflow of programming and formal mathematics is still largely inspired by legacy software development (emacs, make, svn, diffs...)

A paradoxical situation

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We have powerful tools to mechanize the metatheory of (proof) languages

. . . And yet,

Workflow of programming and formal mathematics is still largely inspired by legacy software development (emacs, make, svn, diffs. . .)

Isn't it time to make these tools metatheory-aware?

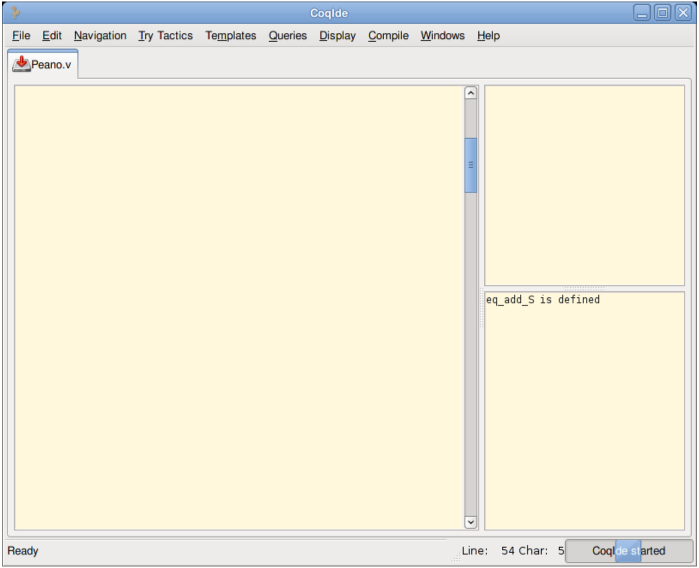
Incrementality in programming & proof languages

Q : Do you spend more time *writing* code or *editing* code?

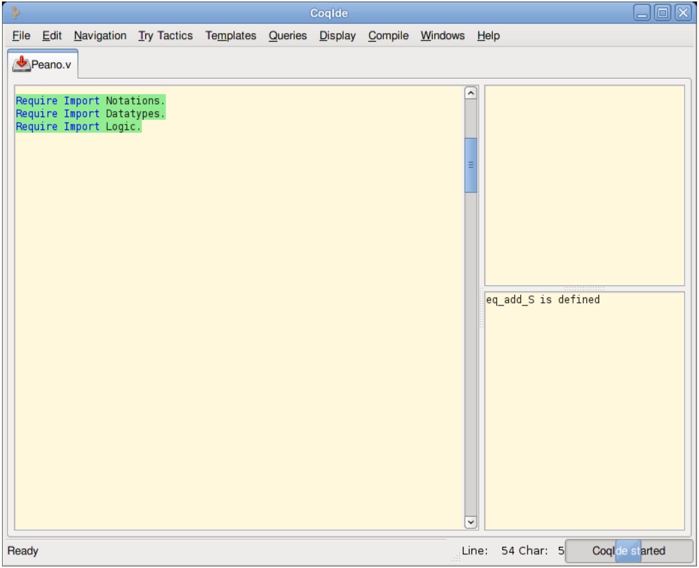
Today, we use:

- separate compilation
- dependency management
- version control on the scripts
- interactive toplevel with rollback (Coq)

Incrementality in programming & proof languages



Incrementality in programming & proof languages



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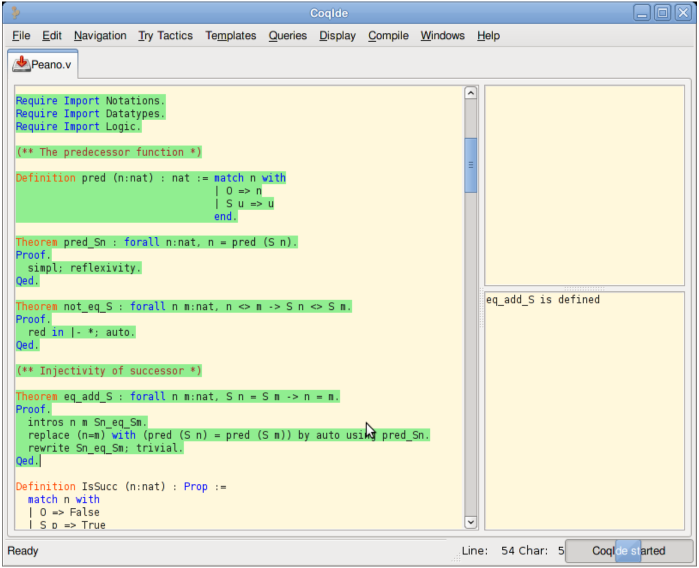

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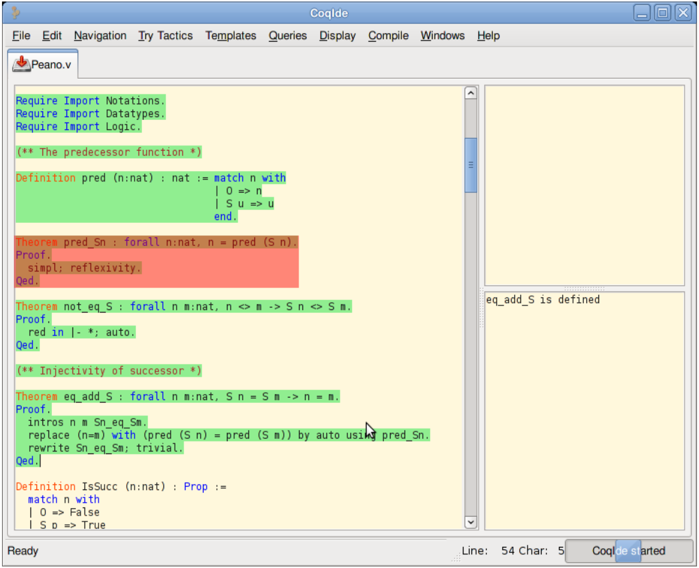

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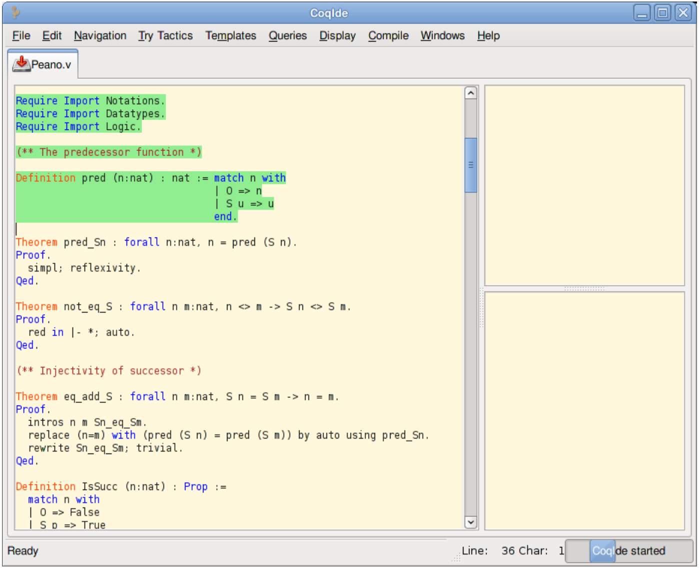

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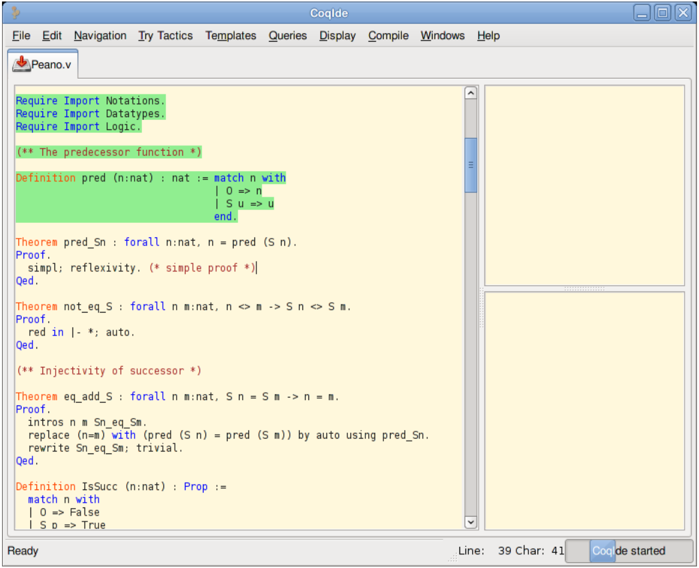

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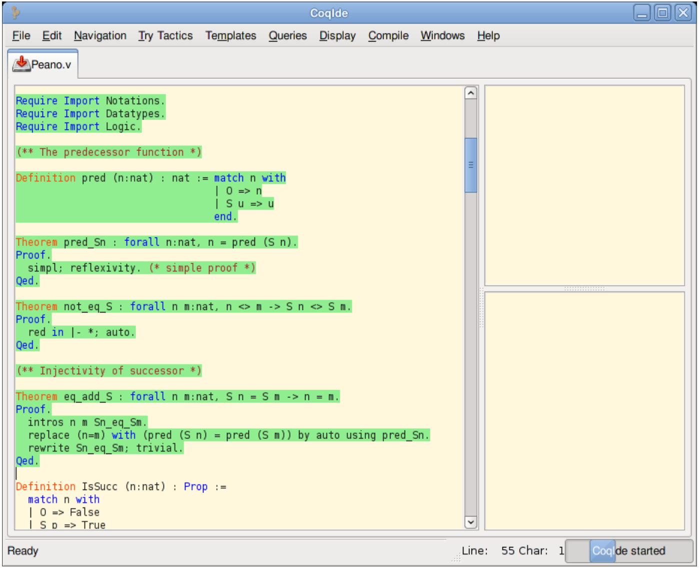

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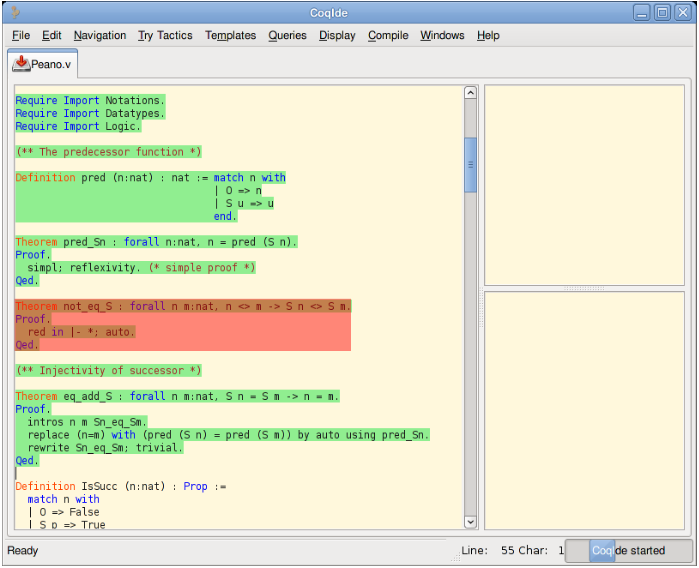

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- Edition should be possible anywhere
- The impact of changes visible “in real time”
- No need for separate compilation, dependency management

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Types are good witnesses of this impact

Applications

- non-linear user interaction
- tactic languages
- type-directed programming
- typed version control systems

Menu

Menu

A logical framework for incremental type-checking

Yes, we're speaking about (any) typed language.

A type-checker

$$\textbf{val}\ \mathsf{check} : \mathsf{env} \to \mathsf{term} \to \mathsf{types} \to \mathsf{bool}$$

- builds and checks the derivation (on the stack)
- conscientiously discards it

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A type-checker

```
val check : env → term → types → bool
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$$
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val check : env $\to$ term $\to$ types $\to$ bool

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A logical framework for incremental type-checking

Goal Type-check a large derivation taking advantage of the knowledge from type-checking previous versions

Idea Remember all derivations!

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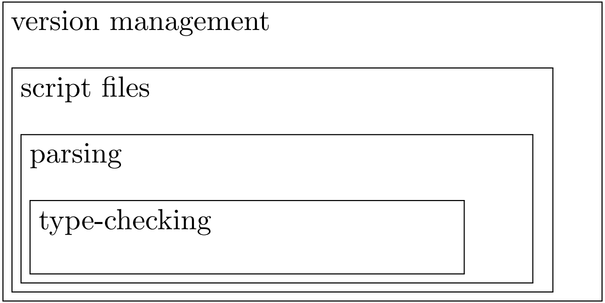
Idea Remember all derivations!

Q Do we really need faster type-checkers?
A Yes, since we implemented these ad-hoc fixes.

The big picture



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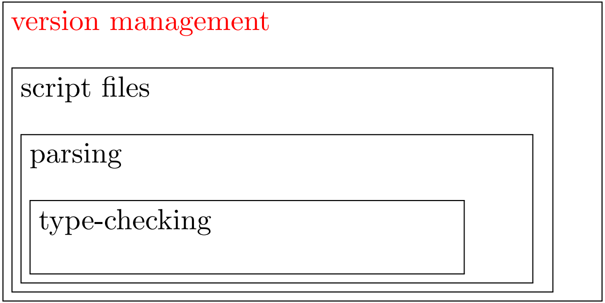

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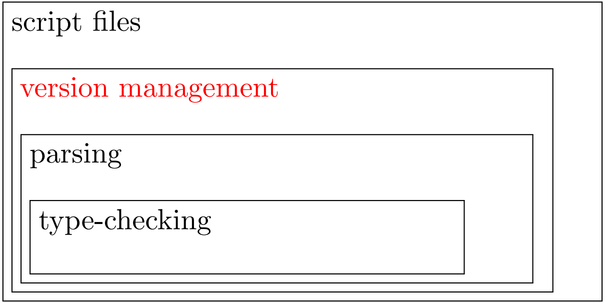

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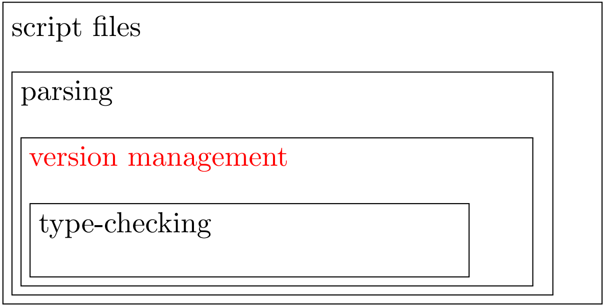
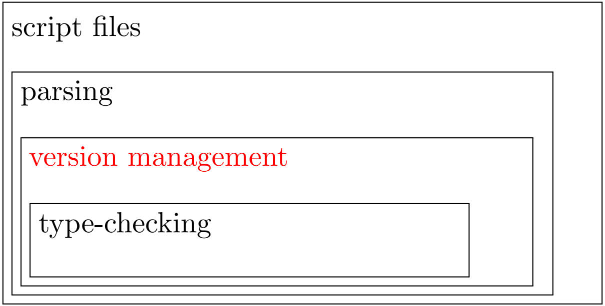


- AST representation

The big picture



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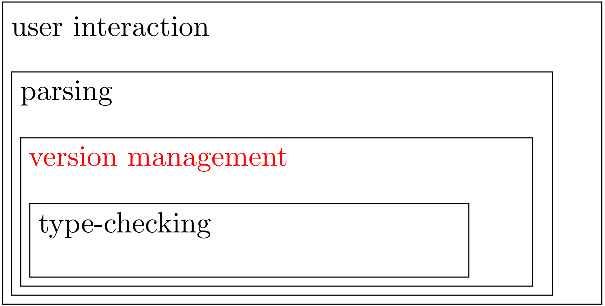

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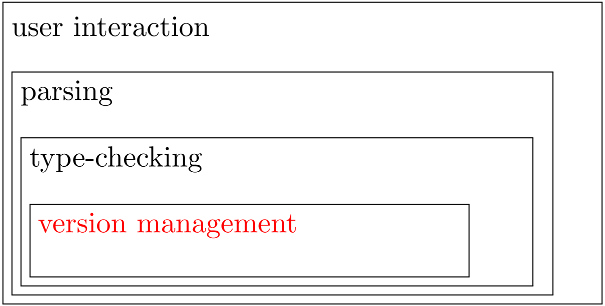

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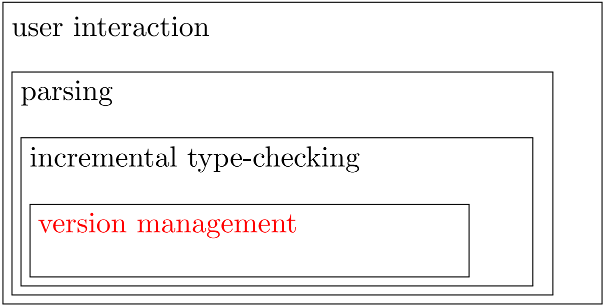

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Menu

Memoization maybe?

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let rec check env t a =
  match t with
  | ... → ... false
  | ... → ... true

and infer env t =
  match t with
  | ... → ... None
  | ... → ... Some a
```

Memoization maybe?

```
let table = ref ([] : environ × term × types) in
let rec check env t a =
  if List.mem (env,t,a) !table then true else
    match t with
    | ... → ... false
    | ... → ... table := (env,t,a)::!table; true
and infer env t =
  try List.assoc (env,t) !table with Not_found →
    match t with
    | ... → ... None
    | ... → ... table := (env,t,a)::!table; Some a
```

Memoization maybe?

+ lightweight

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+ efficient implementation

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− imperative

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  What does it mean logically?

$$\frac{J \in \Gamma}{\Gamma \vdash J \ \mathsf{wf} \Rightarrow \Gamma}$$

$$\frac{\Gamma_1 \vdash J_1 \ \mathsf{wf} \Rightarrow \Gamma_2 \qquad \ldots \qquad \Gamma_{n-1}[J_{n-1}] \vdash J_n \ \mathsf{wf} \Rightarrow \Gamma_n}{\Gamma_1 \vdash J \ \mathsf{wf} \Rightarrow \Gamma_n[J_n][J]}$$

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- – still no trace of the derivation

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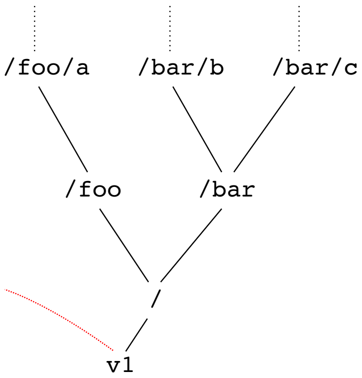
- − syntactic comparison
- − still no trace of the derivation
- \+ gives good reasons to go on

A popular storage model for repositories



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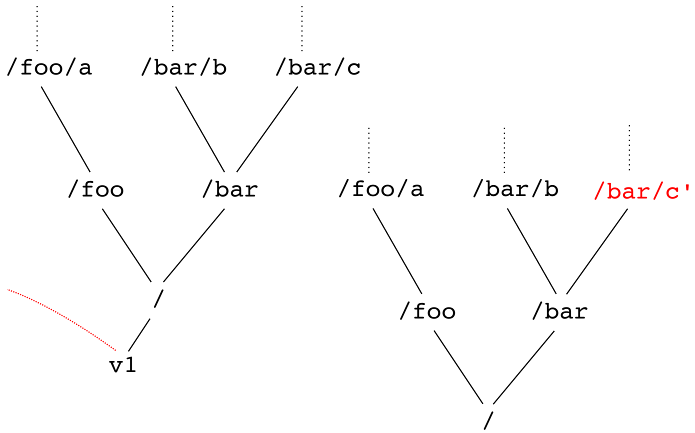

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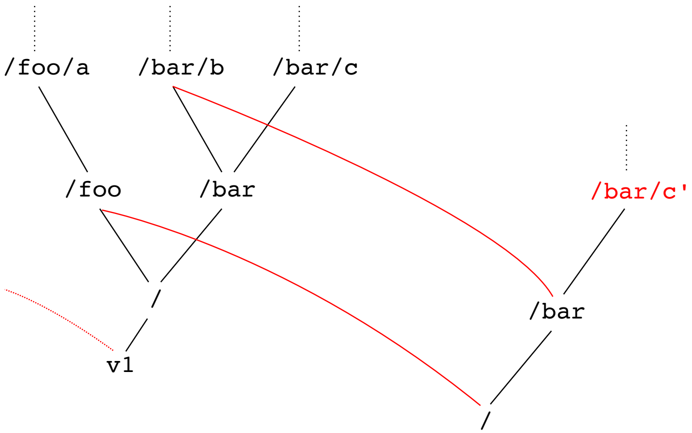

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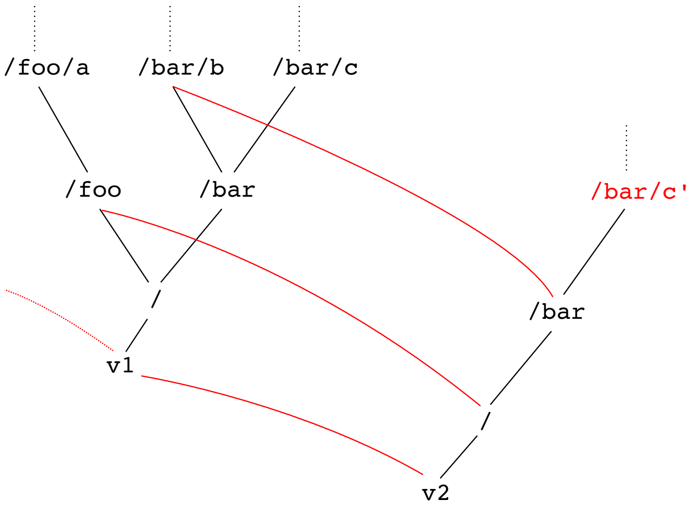

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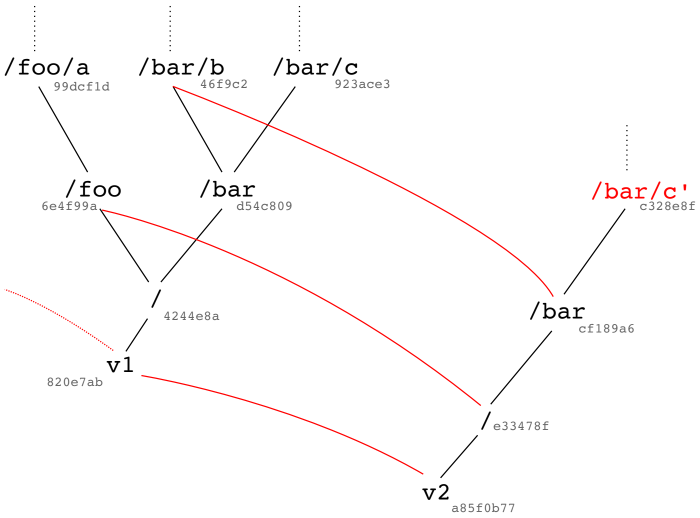

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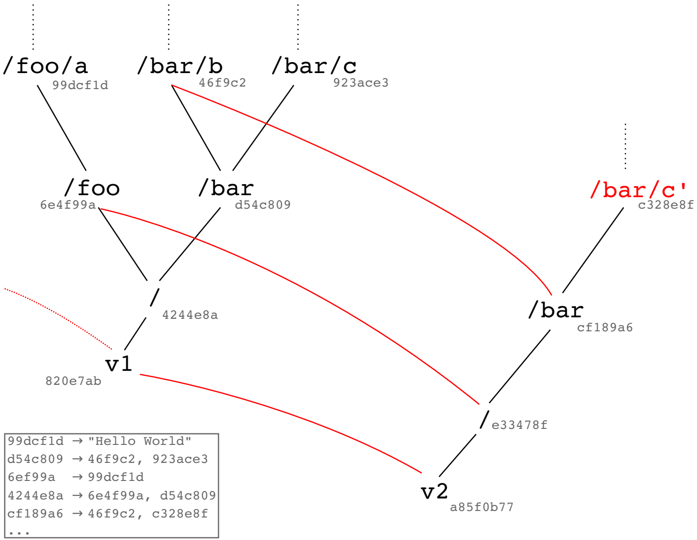

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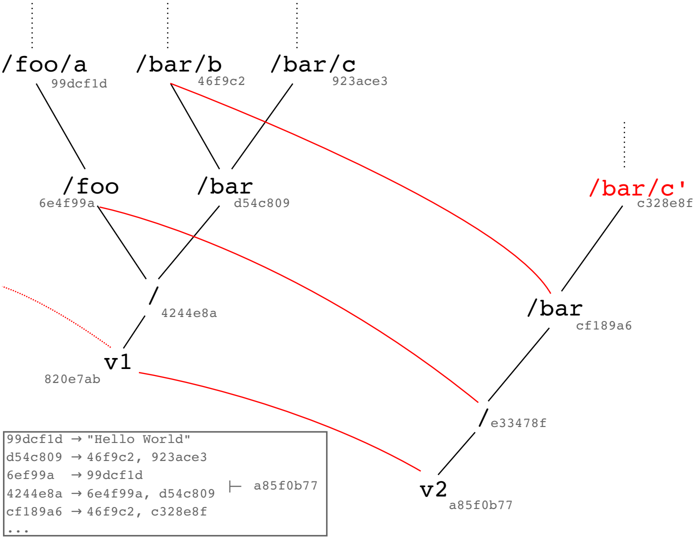

A popular storage model for repositories

The repository R is a pair (Δ, x):

$$\Delta : x \mapsto (\mathsf{Commit}\ (x \times y) \mid \mathsf{Tree}\ \vec{x} \mid \mathsf{Blob}\ string)$$

with the invariants:

- if $(x, \mathsf{Commit}\ (y, z)) \in \Delta$ then
  - $(y, \mathsf{Tree}\ t) \in \Delta$
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- if $(x, \mathsf{Tree}(\vec{y})) \in \Delta$ then
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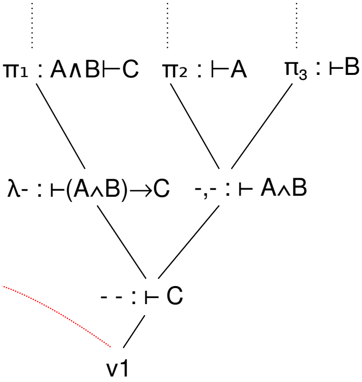
Let's do the same with *proofs*

A *typed* repository of proofs



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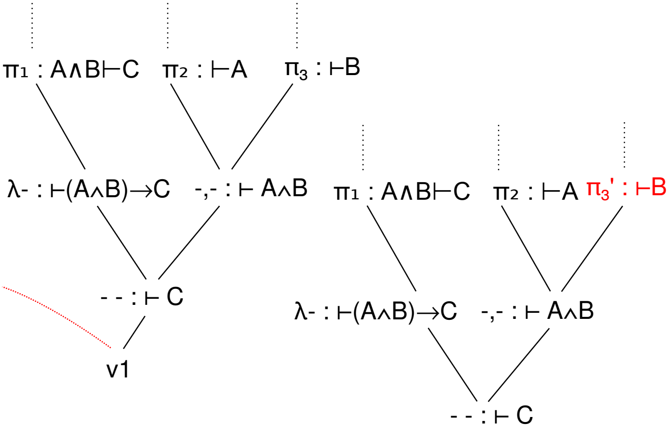

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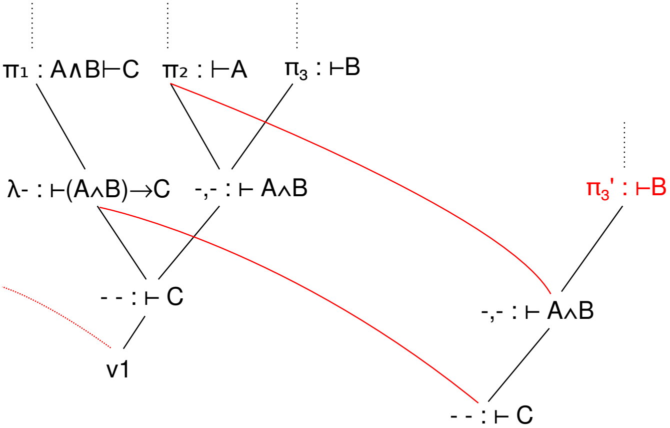

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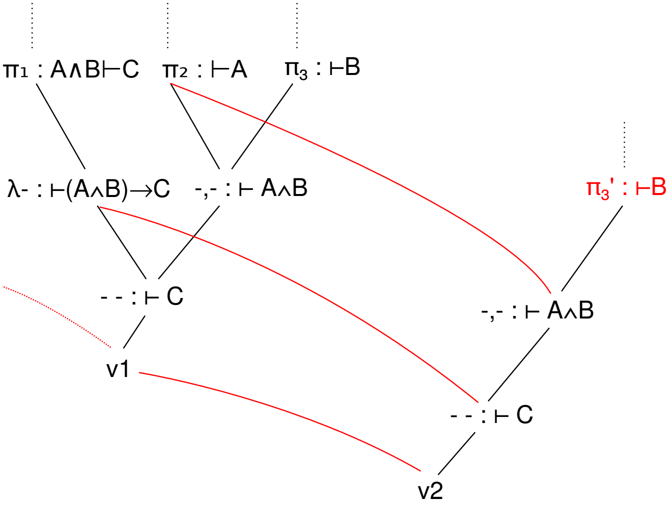

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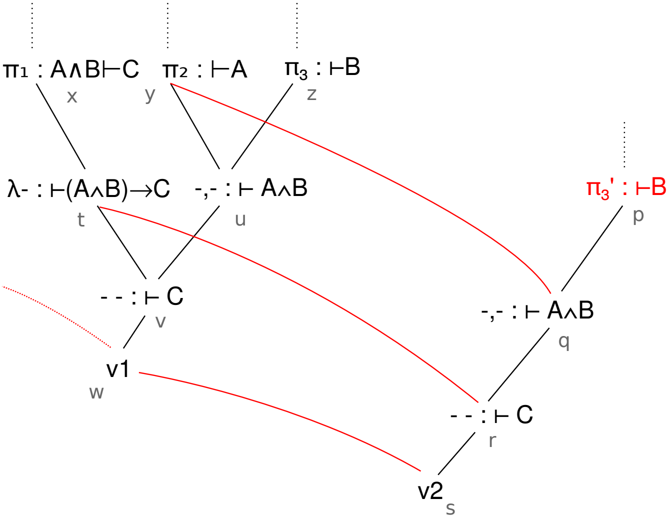

A *typed* repository of proofs

$$
\begin{aligned}
x &= \dots \ : A \wedge B \vdash C \\
y &= \dots \ : \vdash A \\
z &= \dots \ : \vdash B \\
t &= \lambda a^{A \wedge B} \cdot x : \vdash A \wedge B \to C \\
u &= (y, z) : \vdash A \wedge B \\
v &= t\ u : \vdash C \\
w &= \mathsf{Commit}(v, w1) : \mathsf{Version}
\end{aligned}
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A *typed* repository of proofs

```
let x = ... : is (cons (conj A B) nil) C in
  let y = ... : is nil A in
    let z = ... : is nil B in
      let t = lam (conj A B) x : is nil (arr (conj A B) C) in
        let u = pair y z : is nil (conj A B) in
          let v = app t u : is nil C in
            let w = commit v w1 : version in
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A *typed* repository of proofs

```
...
val is : env → prop → type
val conj : prop → prop → prop
val pair : is α β → is α γ → is α (conj β γ)
val version : type
val commit : is nil C → version → version
...
```

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LF [Harper et al. 1992] provides a way to represent and validate syntax, rules and proofs by means of a typed λ-calculus. But we need a little bit more:

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1. definitions / explicit substitutions
2. type annotations on application spines
3. fully applied constants / η-long NF
4. Naming of all application spines / A-normal form
   (= construction of syntax/proofs)

Positionality

$R =$

```
let x = ... : is (cons (conj A B) nil) C in
  let y = ... : is nil A in
    let z = ... : is nil B in
      let t = lam (conj A B) x : is nil (arr (conj A B) C) in
        let u = pair y z : is nil (conj A B) in
          let v = app t u : is nil C in
            let w = commit v w1 : version in
              w
```

Positionality

$R =$

```
let x = ... : is (cons (conj A B) nil) C in
  let y = ... : is nil A in
    let z = ... : is nil B in
      let t = lam (conj A B) x : is nil (arr (conj A B) C) in
        let u = pair y z : is nil (conj A B) in
          let v = app t u : is nil C in
            let w = commit v w1 : version in
              w
```

- Expose the *head* of the term

Positionality

$R =$

```
let x = ... : is (cons (conj A B) nil) C in
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        let u = pair y z : is nil (conj A B) in
          let v = app t u : is nil C in
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              w
```

- Expose the *head* of the term

$$(\lambda x.\lambda y.T)\ U\ V$$

Positionality

$R =$

```
let x = ... : is (cons (conj A B) nil) C in
  let y = ... : is nil A in
    let z = ... : is nil B in
      let t = lam (conj A B) x : is nil (arr (conj A B) C) in
        let u = pair y z : is nil (conj A B) in
          let v = app t u : is nil C in
            let w = commit v w1 : version in
              w
```

- Expose the *head* of the term

$$(\lambda x.\lambda y.T)\ U\ V$$

- Abstract from the *positions* of the binders
  (from inside and from outside)

Menu

Presentation (of the ongoing formalization)

- alternative syntax for LF
- a datastructure of LF derivations
- the repository storage model

Motto: *Take control of the environment*

$$\lambda_{\text{LF}} \xrightarrow[\text{annotation}]{\text{LJ-style}} \text{XLF} \xrightarrow[\text{forget positions}]{\text{heads exposed}} \text{NLF}$$

From LF to XLF

$$
\begin{aligned}
K &::= \Pi x^A \cdot K \mid * \\
A &::= \Pi x^A \cdot A \mid A\ t \mid a \\
t &::= \lambda x^A \cdot t \mid \mathsf{let}\ x = t\ \mathsf{in}\ t \mid t\ t \mid x \mid c \\
\Gamma &::= \cdot \mid \Gamma\,[x : A] \mid \Gamma\,[x = t] \\
\Sigma &::= \cdot \mid \Sigma\,[c : A] \mid \Sigma\,[a : K]
\end{aligned}
$$

- start from standard λ_{LF} with definitions

From LF to XLF

$$
\begin{aligned}
K &::= \Pi x^A \cdot K \mid * \\
A &::= \Pi x^A \cdot A \mid A[l] \mid a[l] \\
t &::= \lambda x^A \cdot t \mid \mathsf{let}\ x = t\ \mathsf{in}\ t \mid t[l] \mid x[l] \mid c[l] \\
l &::= \cdot \mid t; l \\
\Gamma &::= \cdot \mid \Gamma\,[x : A] \mid \Gamma\,[x = t] \\
\Sigma &::= \cdot \mid \Sigma\,[c : A] \mid \Sigma\,[a : K]
\end{aligned}
$$

- start from standard λ_{LF} with definitions
- sequent calculus-like applications ($\bar{\lambda}$)

From LF to XLF

$$
\begin{aligned}
K &::= \Pi x^A \cdot K \mid * \\
A &::= \Pi x^A \cdot A \mid A[l] : K \mid a[l] : K \\
t &::= \lambda x^A \cdot t \mid \mathsf{let}\ x = t\ \mathsf{in}\ t \mid t[l] : A \mid x[l] : A \mid c[l] : A \\
l &::= \cdot \mid t; l \\
\Gamma &::= \cdot \mid \Gamma\,[x : A] \mid \Gamma\,[x = t] \\
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$$

- start from standard λ_{LF} with definitions
- sequent calculus-like applications ($\bar{\lambda}$)
- type annotations on application spines

From LF to XLF

$$
\begin{array}{rcl}
K & ::= & \Pi x^A \cdot K \mid * \\
A & ::= & \Pi x^A \cdot A \mid A[l] : K \mid a[l] : K \\
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\end{array}
$$

- start from standard λ_{LF} with definitions
- sequent calculus-like applications ($\bar{\lambda}$)
- type annotations on application spines

$$
\frac{\Gamma \vdash t : A \qquad \Gamma, B\{x/t\} \vdash l : C}{\Gamma, \Pi x^A \cdot B \vdash t; l : C}
$$

From LF to XLF

$$
\begin{array}{rcl}
K & ::= & \Pi x^A \cdot K \mid * \\
A & ::= & \Pi x^A \cdot A \mid A[l] : K \mid a[l] : K \\
t & ::= & \lambda x^A \cdot t \mid \mathsf{let}\ x = t\ \mathsf{in}\ t \mid t[l] : A \mid x[l] : A \mid c[l] : A \\
l & ::= & \cdot \mid t; l \\
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\Sigma & ::= & \cdot \mid \Sigma\,[c : A] \mid \Sigma\,[a : K]
\end{array}
$$

- start from standard λ_{LF} with definitions
- sequent calculus-like applications ($\bar{\lambda}$)
- type annotations on application spines

$$
\frac{\Gamma \vdash t : A \qquad \Gamma\,[x = t]\,, B \vdash l : C}{\Gamma, \Pi x^A \cdot B \vdash t; l : C}
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From LF to XLF

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K &::= \Pi x^A \cdot K \mid * \\
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\end{aligned}
$$

- start from standard λ_{LF} with definitions
- sequent calculus-like applications ($\bar{\lambda}$)
- type annotations on application spines
- named arguments

$$
\frac{\Gamma \vdash t : A \qquad \Gamma\,[x = t]\,, B \vdash l : C}{\Gamma, \Pi x^A \cdot B \vdash x = t; l : C}
$$

From LF to XLF

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K & ::= & \Pi x^A \cdot K \mid * \\
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\end{array}
$$

- start from standard λ_{LF} with definitions
- sequent calculus-like applications ($\bar{\lambda}$)
- type annotations on application spines
- named arguments

$$
\mathsf{FV}(t[l] : A) = \mathsf{FV}(t) \cup \mathsf{FV}(l) \cup (\mathsf{FV}(A) - \mathsf{FV}(l))
$$
$$
\mathsf{FV}(x = t; l) = \mathsf{FV}(t) \cup (\mathsf{FV}(l) - \{x\})
$$

XLF: Properties

- LJ-style application
- type annotation on application spines
- named arguments (labels)

Lemma (Conservativity)

- $\Gamma \vdash_{\mathrm{LF}} K \ \mathsf{kind} \quad \textit{iff} \quad |\Gamma| \vdash_{\mathrm{XLF}} |K| \ \mathsf{kind}$
- $\Gamma \vdash_{\mathrm{LF}} A \ \mathsf{type} \quad \textit{iff} \quad |\Gamma| \vdash_{\mathrm{XLF}} |A| \ \mathsf{type}$
- $\Gamma \vdash_{\mathrm{LF}} t : A \quad \textit{iff} \quad |\Gamma| \vdash_{\mathrm{XLF}} |t| : |A|$

From XLF to NLF

$$
\begin{aligned}
K &::= \Pi x^A \cdot K \mid * \\
A &::= \Pi x^A \cdot A \mid A[l] : K \mid a[l] : K \\
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\end{aligned}
$$

- start from XLF

From XLF to NLF

$$
\begin{aligned}
K &::= \Pi x^A \cdot K \mid h_K \\
h_K &::= * \\
A &::= \Pi x^A \cdot A \mid A[l] : K \mid a[l] : K \\
t &::= \lambda x^A \cdot t \mid \mathsf{let}\ x = t\ \mathsf{in}\ t \mid t[l] : A \mid x[l] : A \mid c[l] : A \\
l &::= \cdot \mid x = t; l \\
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\end{aligned}
$$

- start from XLF
- isolate heads (non-binders)

From XLF to NLF

$$
\begin{aligned}
K &::= \Pi x^A \cdot K \mid h_K \\
h_K &::= * \\
A &::= \Pi x^A \cdot A \mid h_A \\
h_A &::= A[l] : K \mid a[l] : K \\
t &::= \lambda x^A \cdot t \mid \mathsf{let}\ x = t\ \mathsf{in}\ t \mid t[l] : A \mid x[l] : A \mid c[l] : A \\
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\end{array}
$$

- start from XLF
- isolate heads (non-binders)
- enforce η-long forms by annotating with heads

From XLF to NLF

$$
\begin{array}{rcl}
K & ::= & \Pi x^A \cdot K \mid h_K \\
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\end{array}
$$

- start from XLF
- isolate heads (non-binders)
- enforce η-long forms by annotating with heads

$$t[l] : \Pi x^A \cdot B \quad \longrightarrow_\eta \quad \lambda x^A \cdot (t[x = x; l] : B) \qquad x \notin \mathsf{FV}(t)$$

From XLF to NLF

$$
\begin{array}{rcl}
K & ::= & \Pi x^A \cdot K \mid h_K \\
h_K & ::= & * \\
A & ::= & \Pi x^A \cdot A \mid h_A \\
h_A & ::= & A[l] : h_K \mid a[l] : h_K \\
t & ::= & \Gamma \vdash h_t \\
h_t & ::= & t[l] : h_A \mid x[l] : h_A \mid c[l] : h_A \\
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$$

- start from XLF
- isolate heads (non-binders)
- enforce η-long forms by annotating with heads
- factorize binders and environments

From XLF to NLF

$$
\begin{array}{rcl}
K & ::= & \Pi x^A \cdot K \mid h_K \\
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A & ::= & \Gamma \vdash h_A \\
h_A & ::= & A[l] : h_K \mid a[l] : h_K \\
t & ::= & \Gamma \vdash h_t \\
h_t & ::= & t[l] : h_A \mid x[l] : h_A \mid c[l] : h_A \\
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$$

- start from XLF
- isolate heads (non-binders)
- enforce η-long forms by annotating with heads
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From XLF to NLF

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h_t & ::= & t[l] : h_A \mid x[l] : h_A \mid c[l] : h_A \\
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\Gamma & ::= & \cdot \mid \Gamma\,[x : A] \mid \Gamma\,[x = t] \\
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\end{array}
$$

- start from XLF
- isolate heads (non-binders)
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t & ::= & \Gamma \vdash h_t \\
h_t & ::= & t[l] : h_A \mid x[l] : h_A \mid c[l] : h_A \\
l & ::= & \Gamma \\
\Gamma & ::= & \cdot \mid \Gamma\,[x : A] \ \mid \Gamma\,[x = t] \\
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\end{array}
$$

- start from XLF
- isolate heads (non-binders)
- enforce η-long forms by annotating with heads
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From XLF to NLF

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\begin{aligned}
K &::= \Gamma \vdash h_k \\
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h_A &::= A[l] : h_K \mid a[l] : h_K \\
t &::= \Gamma \vdash h_t \\
h_t &::= t[l] : h_A \mid x[l] : h_A \mid c[l] : h_A \\
l &::= \Gamma \\
\Gamma &: x \mapsto ([x : A] \mid [x = t]) \\
\Sigma &::= \cdot \mid \Sigma\,[c : A] \mid \Sigma\,[a : K]
\end{aligned}
$$

- start from XLF
- isolate heads (non-binders)
- enforce η-long forms by annotating with heads
- factorize binders and environments
- abstract over environment datastructure (maps)

NLF

Syntax

$$
\begin{array}{rcl}
K & ::= & \Gamma\ \mathsf{kind} \\
A & ::= & \Gamma \vdash h_A\ \mathsf{type} \\
h_A & ::= & a\ \Gamma \\
t & ::= & \Gamma \vdash h_t : h_A \\
h_t & ::= & t\ \Gamma \mid x\ \Gamma \mid c\ \Gamma \\
\Gamma & : & x \mapsto ([x : a] \mid [x = t])
\end{array}
$$

Judgements

- $\Gamma\ \mathsf{kind}$
- $\Gamma \vdash h_A\ \mathsf{type}$
- $\Gamma \vdash h_t : h_A$

NLF

Syntax

$$
\begin{array}{rcl}
K & ::= & \Gamma\ \textsf{kind} \\
A & ::= & \Gamma \vdash h_A\ \textsf{type} \\
h_A & ::= & a\ \Gamma \\
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\end{array}
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Judgements

- K
- A
- t

NLF

Syntax

$$
\begin{array}{rcl}
K & ::= & \Gamma\ \textsf{kind} \\
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h_t & ::= & t\ \Gamma \mid x\ \Gamma \mid c\ \Gamma \\
\Gamma & : & x \mapsto ([x : a] \mid [x = t])
\end{array}
$$

Judgements

- K wf
- A wf
- t wf

NLF

Notations

- "h_A" for "$\vdash h_A$"
- "h_A" for "$\emptyset \vdash h_A \; \mathsf{type}$"
- "a" for "$a \; \emptyset$"

Example

$$\lambda f^{A \to B} \cdot \lambda x^A \cdot f \; x : (A \to B) \to A \to B \qquad \equiv$$

$$[f : [a : A] \vdash B \; \mathsf{type}] \, [x : A] \vdash f \; [a = x] : B$$

Some more examples

$$A : \emptyset \ \text{kind}$$
$$\equiv *$$

$$vec : [len : \mathbb{N}] \ \text{kind}$$
$$\equiv \mathbb{N} \to *$$

$$nil : \ \vdash vec \ [len = \ \vdash 0 : \mathbb{N}] \ \text{type}$$
$$\equiv vec \ 0 : *$$

$$cons : [l : \mathbb{N}] \, [hd : A] \, [tl : \ \vdash vec \ [len = \ \vdash l : \mathbb{N}] \ \text{type}] \vdash vec \ [len = \ \vdash s \ [n = \ \vdash l : \mathbb{N}] : \mathbb{N}] \ \text{type}$$
$$\equiv \Pi l^{\mathbb{N}} \cdot A \to \Pi tl^{vec \ l} \cdot vec \ (s \ l : \mathbb{N}) : *$$

$$fill : [n : \mathbb{N}] \vdash vec \ [len = \ \vdash n : \mathbb{N}] \ \text{type}$$
$$\equiv \Pi n^{\mathbb{N}} \cdot (vec \ n : *)$$

$$empty : [e : vec \ [len = 0]] \ \text{kind}$$
$$\equiv vec \ 0 \to *$$

$$- : \ \vdash empty \ [e = \ \vdash fill \ [n = 0] : vec \ [len = n]] \ \text{type}$$
$$\equiv empty \ (fill \ 0 : vec \ 0)$$

Environments

... double as labeled *directed acyclic graphs* of dependencies:

Definition (environment)

$\Gamma = (V, E)$ directed acyclic where:

- $V \subseteq \mathcal{X} \times (t \uplus A)$ and
- $(x, y) \in E$ (x depends on y) if $y \in \mathsf{FV}(\Gamma(x))$

Definition (lookup)

$$\Gamma(x) : A \quad \text{if} \quad (x, A) \in E$$
$$\Gamma(x) = t \quad \text{if} \quad (x, t) \in E$$

Definition (bind)

$$\Gamma\,[x : A] = (V \cup (x, A), E \cup \{(x, y) \mid y \in \mathsf{FV}(A)\})$$
$$\Gamma\,[x = t] = (V \cup (x, A), E \cup \{(x, y) \mid y \in \mathsf{FV}(t)\})$$

Environments

… double as labeled *directed acyclic graphs* of dependencies:

Definition (decls, defs)

$\mathsf{decls}(\Gamma) = [x_1, \ldots, x_n]$ s.t. $\Gamma(x_i) : A_i$ topologically sorted *wrt.* Γ
$\mathsf{defs}(\Gamma) = [x_1, \ldots, x_n]$ s.t. $\Gamma(x_i) = t_i$ topologically sorted *wrt.* Γ

Definition (merge)

$\Gamma \cdot \Delta = \Gamma \cup \Delta$ s.t.

- if $\Gamma(x) : A$ and $\Gamma(x) = t$ then $\Gamma \cdot \Delta(x) = t$
- undefined otherwise

Reduction

Definition (A-contexts)

$\Delta^* = \{\, [x = x] \mid x \in \mathsf{defs}(\Delta)\}$

$$
\begin{array}{rcll}
\Gamma \vdash (\Delta \vdash h_t : h_A)\ \Xi : \_ & \xrightarrow{\text{``}\beta\text{''}} & \Gamma \cdot \Delta \cdot \Xi \vdash h_t : h_A & \\
\Gamma \vdash c\ \Delta : h_A & \longrightarrow & \Gamma \cdot \Delta \vdash c\ \Delta^* : h_A & \text{if } \Delta \neq \Delta^* \\
\Gamma \vdash c\ \Xi^* : a\ \Delta & \longrightarrow & \Gamma \cdot \Delta \vdash c\ \Xi^* : a\ \Delta^* & \text{if } \Delta \neq \Delta^*
\end{array}
$$

Typing

$$\textsc{Fam}\quad \frac{\Sigma(a) : (\Xi\ \textsf{kind}) \qquad \Gamma \vdash \Delta : \Xi}{\Gamma \vdash a\ \Delta\ \textsf{type}}$$

$$\textsc{Objc}\quad \frac{\Sigma(c) : (\Xi \vdash h_A\ \textsf{type}) \qquad \Gamma \vdash \Delta : \Xi \qquad \Gamma \cdot \Xi \cdot \Delta \vdash h'_A \equiv h_A\ \textsf{type}}{\Gamma \vdash c\ \Delta : h'_A}$$

$$\textsc{Objx}\quad \frac{\Gamma(x) = (\Xi \vdash h_t : h_A) \qquad \Gamma \vdash \Delta : \Xi \qquad \Gamma \cdot \Xi \cdot \Delta \vdash h'_A \equiv h_A\ \textsf{type} \qquad \Gamma \cdot \Xi \cdot \Delta \vdash h_t : h_A}{\Gamma \vdash x\ \Delta : h'_A}$$

$$\textsc{Args}\quad \frac{\forall x \in \textsf{decls}(\Xi) \quad \Delta(x) = (\Delta' \vdash h_t : h_A) \qquad \Xi(x) : (\Xi' \vdash h'_A\ \textsf{type}) \quad \Gamma \cdot \Delta \cdot \Delta' \vdash h_t : h_A \qquad \Gamma \cdot \Xi \cdot \Delta \cdot \Xi' \cdot \Delta' \vdash h'_A \equiv h_A\ \textsf{type}}{\Gamma \vdash \Delta : \Xi}$$

Properties

Translation functions

- $|\cdot|_\Gamma \;:\; K_{\mathrm{LF}} \to \Gamma_{\mathrm{NLF}} \to K_{\mathrm{NLF}}$ option
- $|\cdot|_\Gamma \;:\; A_{\mathrm{LF}} \to \Gamma_{\mathrm{NLF}} \to A_{\mathrm{NLF}}$ option
- $|\cdot|_\Gamma \;:\; t_{\mathrm{LF}} \to \Gamma_{\mathrm{NLF}} \to t_{\mathrm{NLF}}$ option
- $\ldots$ and their inverses $|\cdot|^{-1}$

Conjecture (Conservativity)

- $\vdash_{\mathrm{LF}} K$ kind $\quad$ iff $\quad (|K|_\emptyset)$ wf
- $\vdash_{\mathrm{LF}} A$ type $\quad$ iff $\quad (|A|_\emptyset)$ wf
- $\vdash_{\mathrm{LF}} t : A$ $\quad$ iff $\quad (|t|_\emptyset)$ wf

Menu

Status

```
$ ./gasp init hol.elf
```

Status

```
$ ./gasp init hol.elf

[holtype : kind]
[i : holtype]
[o : holtype]
[arr : [x2 : holtype][x1 : holtype] ⊢ holtype type]

Fatal error: exception Assert_failure("src/NLF.ml", 61, 13)
```

Checkout

```
$ ./gasp checkout v42
```

if

$$t = \Gamma \vdash v_{52} : \mathsf{Version} \quad \text{and} \quad \Gamma(v_{42}) = \mathsf{Commit}(v_{41}, h)$$

then

$$|\Gamma(h)|^{-1}$$

is the LF term representing `v42`

Commit

```
$ ./gasp commit term.elf
```

if

$$t = \Gamma \vdash v_{52} : \mathsf{Version} \quad \text{and} \quad |\texttt{term.elf}|_\Gamma = \Delta \vdash h_t : h_A$$

then

$$\Delta\,[v_{53} = \mathsf{Commit}\ [prev = v_{52}]\,[this = h_t]] \vdash v_{53} : \mathsf{Version}$$

is the new repository

Further work

- still some technical & metatheoretical unknowns
- from derivations to terms (proof search? views?)
- diff on terms or derivations
- type errors handling and recovery
- mimick other operations from VCS (Merge)