A Contextual Account of Staged Computations

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Groupe de Travail Théorie des Types et Réalisabilité
PPS, Univ. Paris Diderot

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(WIP)

Those red apples

⟨Those red apples ⟩ is a noun phrase.

For any noun n,
$\langle$Those red $\sim$(pluraled n)$\rangle$ is a noun phrase.

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is valid a grammar rule.

For some part of speech P, $\langle$For any $\sim(P)$ n,
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For some part of speech P, ⟨For any ∼(P) n,
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This talk is about typing such metalanguages in a *principled* way.

Motivation 1: Macros

In ML, **if** is syntactic sugar:

$$(\textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3) \triangleq (\textbf{match}\ e_1\ \textbf{with}\ \textbf{true} \rightarrow e_2 \mid \textbf{false} \rightarrow e_3)$$

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Problem

We can't define it in CBV:

let `if_ e1 e2 e3` = **match** `e1` **with true** → `e2` | **false** → `e3`

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let `if_ e1 e2 e3` = **match** `e1` **with true** → `e2` | **false** → `e3` **in**
`if_` **false** (`raise` Exit) (`print_string` "Hello")

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let if_ e1 e2 e3 = match e1 with true → e2 | false → e3 in
if_ false (raise Exit) (print_string "Hello");;
Hello Exception: Pervasives.Exit.                  (* fail *)
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```

⇝ How to define syntactic sugar in the language?

Motivation 2: Program specialization

```
let rec pow x n =
  if n = 0 then 1
  else if n mod 2 = 0 then sq (pow x (n/2))
  else x * pow x (n-1)
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For performance, we want to derive specialized programs:

```
let pow13 x = pow x 13              (* a closure of pow *)
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For performance, we want to derive specialized programs:

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let pow13 x = pow x 13                       (* a closure of pow *)
let pow13s x = x * sq (sq (x * sq (x * 1)))  (* 6x faster *)
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⇝ How to specialize a function on a statically known argument?

Motivation 3: Full evaluation

Evaluation stops at λs:

```
let f = fun x → ((fun y → y + 1) x);;
val f : int → int = <fun>
```

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Evaluation stops at λs:

```
let f = fun x → ((fun y → y + 1) x);;
val f : int → int = <fun>
```

Problem

We sometimes need to syntactically compare normal forms.

```
f = fun x → x + 1;;
Exception: Invalid_argument "equal:␣functional␣value".
```

$\rightsquigarrow$ How to evaluate under λs?

One-size-fits-all solution: Staging

A multi-staged functional programming language provides a finer control over evaluation.

One-size-fits-all solution: Staging

A multi-staged functional programming language provides a finer control over evaluation.

The method

Organizes evaluation into ordered stages (level):

- each redex belongs to a stage n,
- redex n fired only if no redex $m < n$

One-size-fits-all solution: Staging

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The abstraction

Generating, grafting and running pieces of code (AST).

Multi-staged languages

Syntax:

$$e \;::=\; \ldots \mid \langle e \rangle \mid {\sim}e \mid \texttt{run}\; e$$

Operational semantics:

$${\sim}\langle v \rangle \longrightarrow v \qquad\qquad \texttt{run}\;\langle e \rangle \longrightarrow e$$

$$C \;::=\; \ldots \mid \langle \ldots {\sim}C \ldots \rangle$$

Examples

Macros

$$\textbf{let}\ \texttt{if\_}\ \texttt{e1}\ \texttt{e2}\ \texttt{e3} = \langle\ \textbf{match}\ {\sim}\texttt{e1}\ \textbf{with true} \rightarrow {\sim}\texttt{e2} \mid \textbf{false} \rightarrow {\sim}\texttt{e3}\ \rangle$$

Examples

Macros

```
let if_ e1 e2 e3 =
  ⟨ match ~e1 with true → ~e2 | false → ~e3 ⟩ ;;

let e = ⟨~(if_ ⟨false⟩ ⟨raise Exit⟩ ⟨print_string "Hello"⟩)⟩
```

Examples

Macros

let if_ e1 e2 e3 =
  ⟨ **match** ~e1 **with true** → ~e2 | **false** → ~e3 ⟩ ;;

let e = ⟨~(if_ ⟨**false**⟩ ⟨raise Exit⟩ ⟨print_string "Hello"⟩)⟩;;
val e =
  ⟨ **match false with**
    | **true** → raise Exit
    | **false** → print_string "Hello" ⟩

Examples

Macros

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let if_ e1 e2 e3 =
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val e =
  ⟨ match false with
     | true  → raise Exit
     | false  → print_string "Hello" ⟩ ;;

run e
```

Examples

Macros

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let if_ e1 e2 e3 =
  ⟨ match ~e1 with true → ~e2 | false → ~e3 ⟩ ;;

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val e =
  ⟨ match false with
     | true  → raise Exit
     | false  → print_string "Hello" ⟩ ;;

run e ;;
Hello − : unit = ()
```

Examples

Program specialization

$$
\begin{array}{l}
\textbf{let rec}\ \texttt{pow x n} = \\
\quad \textbf{if}\ \texttt{n} = 0\ \textbf{then}\ \langle 1 \rangle \\
\quad \textbf{else if}\ \texttt{n}\ \textbf{mod}\ 2 = 0\ \textbf{then}\ \langle \texttt{square}\ \sim(\texttt{pow x (n/2)}) \rangle \\
\quad \textbf{else}\ \langle \sim\texttt{x} * \sim(\texttt{pow x (n-1)}) \rangle
\end{array}
$$

Examples

Program specialization

$$\textbf{let rec}\ \texttt{pow x n} =$$
$$\quad \textbf{if}\ \texttt{n} = 0\ \textbf{then}\ \langle 1 \rangle$$
$$\quad \textbf{else if}\ \texttt{n}\ \textbf{mod}\ 2 = 0\ \textbf{then}\ \langle \texttt{square} \sim(\texttt{pow x (n/2)}) \rangle$$
$$\quad \textbf{else}\ \langle \sim\texttt{x} * \sim(\texttt{pow x (n-1)}) \rangle\ ;;$$

$$\textbf{let}\ \texttt{pow13} = \langle\ \textbf{fun}\ \texttt{x} \to \sim(\texttt{pow}\ \langle \texttt{x} \rangle\ \texttt{13})\ \rangle$$

Examples

Program specialization

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let rec pow x n =
  if n = 0 then ⟨1⟩
  else if n mod 2 = 0 then ⟨square ~(pow x (n/2))⟩
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let pow13 = ⟨ fun x → ~(pow ⟨x⟩ 13) ⟩ ;;
val pow13 =
  ⟨ fun x  → x * square (square (x * square (x * 1)))⟩
```

Examples

Program specialization

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run pow13 2
```

Examples

Program specialization

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let rec pow x n =
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val pow13 =
  ⟨ fun x  → x * square (square (x * square (x * 1)))⟩

run pow13 2 ;;
− : int = 8192
```

Examples

Full evaluation

$$\textbf{let}\ \ \mathtt{e} = \langle \textbf{fun}\ \mathrm{x} \rightarrow {\sim}((\textbf{fun}\ \mathrm{y} \rightarrow \langle\ {\sim}\mathrm{y} + 1\ \rangle)\ \langle \mathrm{x} \rangle) \rangle$$

Examples

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$\textbf{val}\ \ \texttt{e} = \langle \textbf{fun}\ \mathrm{x}\ \ \rightarrow \mathrm{x} + 1 \rangle$

Examples

Full evaluation

$$\textbf{let}\;\; \mathtt{e} = \langle \textbf{fun}\; \mathtt{x} \rightarrow \sim((\textbf{fun}\; \mathtt{y} \rightarrow \langle\; \sim\mathtt{y} + 1\; \rangle)\; \langle \mathtt{x} \rangle)\rangle\;;;$$
$$\textbf{val}\;\; \mathtt{e} = \langle \textbf{fun}\; \mathtt{x}\;\; \rightarrow \mathtt{x} + 1\rangle$$

```
run e 42
```

Examples

Full evaluation

$\textbf{let}\ \texttt{e} = \langle \textbf{fun}\ \mathrm{x} \rightarrow \sim((\textbf{fun}\ \mathrm{y} \rightarrow \langle\ \sim\mathrm{y} + 1\ \rangle)\ \langle \mathrm{x} \rangle) \rangle\ ;;$

$\textbf{val}\ \texttt{e} = \langle \textbf{fun}\ \mathrm{x}\ \rightarrow \mathrm{x} + 1 \rangle$

$\texttt{run}\ \texttt{e}\ 42\ ;;$

$-: \texttt{int} = 43$

Examples

Full evaluation

$$\textbf{let}\ \ \mathtt{e} = \langle\textbf{fun}\ \mathrm{x} \rightarrow \sim((\textbf{fun}\ \mathrm{y} \rightarrow \langle\ \sim\mathrm{y} + 1\ \rangle)\ \langle\mathrm{x}\rangle)\rangle\ ;;$$
$$\textbf{val}\ \ \mathtt{e} = \langle\textbf{fun}\ \mathrm{x}\ \ \rightarrow \mathrm{x} + 1\rangle$$

$$\mathtt{run}\ \mathtt{e}\ 42\ ;;$$
$$-:\ \mathtt{int} = 43$$

Remark

We must now ensure:

- lexical scoping (variables used in their binding context...)
- congruence (...at their binding stage) = "staged lexical scoping"
  <u>ex</u>: $\not\vdash \langle\ \textbf{fun}\ \mathrm{x} \rightarrow \sim(\mathrm{x} + 1)\ \rangle$
- evaluation of closed code
  <u>ex</u>: $\not\vdash \langle\ \textbf{fun}\ \mathrm{x} \rightarrow \sim(\mathtt{run}\ \langle\mathrm{x}\rangle)\ \rangle$

A type system for staged computations?

$\lambda\Box$ (Davies & Pfenning, 1996)

$\Delta; \Gamma \vdash M : A$

S4 "necessarily" modality $\Box A$ = type of *closed* code of type A

(*e.g.* $\Box(\mathsf{nat} \to \mathsf{nat})$)

ensures safe evaluation ($\texttt{run} : \Box A \to A$)

but only closed code

A type system for staged computations?

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$\lambda^{\bigcirc}$ (Davies, 1996) $\quad \boxed{\Gamma \vdash^n M : A}$

LTL "next" modality $\bigcirc A$ = type of *open* code

(*e.g.* $\bigcirc(\mathsf{nat} \to \mathsf{nat})$)

variables indexed by the current stage index n

ensures staged lexical scoping

but no safe code evaluation

A type system for staged computations?

$\lambda^{\square}$ (Davies & Pfenning, 1996) $\Delta;\Gamma \vdash M : A$

S4 "necessarily" modality $\square A$ = type of *closed* code of type A

(*e.g.* $\square(\mathsf{nat} \to \mathsf{nat})$)

ensures safe evaluation ($\texttt{run} : \square A \to A$)

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but no safe code evaluation

λ^{α} (Taha & Nielsen, 2003) $\Gamma \vdash^{\bar{\alpha}} M : A$

generalization of $\lambda^{\bigcirc}$ where stages are *named*

and can be *quantified over* (*e.g.* $\forall\alpha.\langle\mathsf{nat} \to \mathsf{nat}\rangle^{\alpha}$)

ensures safe evaluation & open code

but unclear logical foundation

Outline

In this talk

Close the gap between these systems:

- design a system λ^{ctx}
- show that it embeds them all

Contents

✓ Multi-staged programming by example
- Environment classifiers ($\lambda\bigcirc$, λ^{α})
- Contextual types ($\lambda\Box$, λ^{ctx})
- Translating λ^{α} to λ^{ctx}
- Summary and horizons

The λ-calculus $\lambda^{\rightarrow}$

(Church, 1940)

$$T, U ::= p \mid T \rightarrow U$$

$$E, F ::= x \mid \lambda x.E \mid E F$$

$$\Xi ::= \cdot \mid \Xi, x : T$$

$$\boxed{\Xi \vdash E : T}$$

VAR

$$\frac{(x : T) \in \Xi}{\Xi \vdash x : T}$$

LAM

$$\frac{\Xi, x : T \vdash E : U}{\Xi \vdash \lambda x.E : T \rightarrow U}$$

The temporal λ-calculus $\lambda\bigcirc$

(Davies, 1995)

$$T, U ::= p \mid T \to U \mid \bigcirc T$$
$$E, F ::= x \mid \lambda x. E \mid E\, F \mid \langle E \rangle \mid {\sim}E$$
$$\Xi ::= \cdot \mid \Xi, x :^n T$$

$$\boxed{\Xi \vdash^n E : T}$$

$$\textsc{Var}\ \frac{(x :^n T) \in \Xi}{\Xi \vdash^n x : T} \qquad \textsc{Lam}\ \frac{\Xi, x :^n T \vdash^n E : U}{\Xi \vdash^n \lambda x. E : T \to U} \qquad \textsc{Quote}\ \frac{\Xi \vdash^{n+1} E : T}{\Xi \vdash^n \langle E \rangle : \bigcirc T}$$

$$\textsc{Unquote}\ \frac{\Xi \vdash^n E : \bigcirc T}{\Xi \vdash^{n+1} {\sim}E : T}$$

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- $\texttt{run} : \bigcirc A \to A$ is unsafe
  ex: $\vdash \langle \lambda x. {\sim}(\texttt{run}\ \langle x \rangle) \rangle : \bigcirc(\bigcirc A \to A)$ but gets stuck

The environment classifiers λ-calculus λ^{α}

(Taha & Nielsen, 2003)

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- `run` $: \bigcirc A \to A$ is unsafe
  ex: $\vdash \langle \lambda x. {\sim}(\texttt{run}\ \langle x \rangle) \rangle : \bigcirc(\bigcirc A \to A)$ but gets stuck

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- $\texttt{run} : (\forall \alpha. \langle A \rangle^\alpha) \to \forall \alpha. A$ is safe
  ex: $\nvdash \langle \lambda x. {\sim}(\texttt{run}\ \langle x \rangle^\alpha) \rangle^\alpha$

The environment classifiers λ-calculus λ^{α}

Running code

$\texttt{run} : (\forall\alpha. \langle A\rangle^{\alpha}) \to \forall\alpha.A$

The environment classifiers λ-calculus λ^α

Running code

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Example (Two-level η-expansion)

$\lambda f.\langle\lambda x.\sim(f\,\langle x\rangle)\rangle : (\bigcirc p \to \bigcirc q) \to \bigcirc(p \to q)$

The environment classifiers λ-calculus λ^α

Running code

$\texttt{run} : (\forall\alpha.\langle A\rangle^\alpha) \to \forall\alpha.A$

Example (Two-level η-expansion)

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The environment classifiers λ-calculus λ^α

Running code

$\texttt{run} : (\forall \alpha . \langle A \rangle^\alpha) \to \forall \alpha . A$

Example (Two-level η-expansion)

$\lambda f . \Lambda \alpha . \langle \lambda x . \sim(f \, \alpha \, \langle x \rangle^\alpha) \rangle^\alpha : (\forall \alpha . \langle p \rangle^\alpha \to \langle q \rangle^\alpha) \to \forall \alpha . \langle p \to q \rangle^\alpha$

The environment classifiers λ-calculus λ^α

Running code

$\texttt{run} : (\forall\alpha.\langle A\rangle^\alpha) \to \forall\alpha.A$

Example (Two-level η-expansion)

$\lambda f.\Lambda\alpha.\langle\lambda x.\sim(f\,\alpha\,\langle x\rangle^\alpha)\rangle^\alpha : (\forall\alpha.\langle p\rangle^\alpha \to \langle q\rangle^\alpha) \to \forall\alpha.\langle p \to q\rangle^\alpha$

Issues

- what is the logical meaning of λ^α?
- what do α range over?
- complex operational semantics
  - syntax of value is context-sensitive (no BNF)
    $V^0 ::= \lambda x.V^0 \mid \langle V^1\rangle^\alpha$
  - 14 big-step rules

The λ-calculus $\lambda^{\to}$

(Church, 1940)

$$A, B ::= p \mid A \to B$$
$$M, N ::= x \mid \lambda x. M \mid M\, N$$

$$\boxed{\quad \Gamma \vdash M : A}$$

The modal λ-calculus λ□

(Davies & Pfenning, 1995)

$$A, B ::= p \mid A \to B \mid \Box A$$

$$M, N ::= x \mid \lambda x.\, M \mid M\, N \mid [M] \mid \text{let box } u = M \text{ in } N \mid u$$

$$\boxed{\Delta; \Gamma \vdash M : A}$$

$$\textsc{Box}\quad \frac{\Delta;\ \cdot \vdash M : A}{\Delta; \Gamma \vdash [M] : \Box A}$$

$$\textsc{LetBox}\quad \frac{\Delta; \Gamma \vdash M : \Box A \qquad \Delta, u :: \Box A\ ; \Gamma \vdash N : C}{\Delta; \Gamma \vdash \text{let box } u = M \text{ in } N : C}$$

$$\textsc{Meta}\quad \frac{u :: \Box A \in \Delta}{\Delta; \Gamma \vdash u : A}$$

The contextual λ-calculus λ_E^{ctx}

(Nanevski, Pfenning & Pientka, 2008)

$$A, B ::= p \mid A \to B \mid [\Psi . A]$$

$$M, N ::= x \mid \lambda x. M \mid M\, N \mid [\Psi . M] \mid \text{let box } u = M \text{ in } N \mid u\{\sigma\}$$

$\boxed{\Delta; \Gamma \vdash M : A}$

$$\textsc{Box}\ \frac{\Delta; \Psi \vdash M : A}{\Delta; \Gamma \vdash [\Psi . M] : [\Psi . A]}$$

$$\textsc{LetBox}\ \frac{\Delta; \Gamma \vdash M : [\Psi . A] \qquad \Delta, u :: [\Psi . A]; \Gamma \vdash N : C}{\Delta; \Gamma \vdash \text{let box } u = M \text{ in } N : C}$$

$$\textsc{Meta}\ \frac{u :: [\Psi . A] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Psi}{\Delta; \Gamma \vdash u\{\sigma\} : A}$$

The contextual λ-calculus with first-class envs. λ_E^{ctx}

$$A, B ::= p \mid A \to B \mid [\Psi.A] \mid \forall \alpha.A$$

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A, B &::= p \mid A \to B \mid [\Psi.A] \mid \forall \alpha.A \\
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\Gamma, \Psi &::= \alpha \mid \Gamma, x : A \\
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\end{aligned}
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\textsc{Box}\;\frac{\Delta; \Psi \vdash M : A}{\Delta; \Gamma \vdash [\Psi.M] : [\Psi.A]}
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$$

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A, B &::= p \mid A \to B \mid [\Psi . A] \mid \forall \alpha . A \\
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\Gamma, \Psi &::= \alpha \mid \Gamma, x : A \\
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\end{aligned}
$$

Example

$$\lambda f . \Lambda \alpha . \mathsf{let\,box}\; u = f\,(\alpha, x : p)\,[\alpha, x . x] \;\mathsf{in}\; [\alpha . \lambda x . u\{\mathsf{id}_\alpha, x/x\}]$$

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$$\lambda f. \Lambda \alpha. [\alpha. \lambda x. {\sim}(f\,(\alpha, x : p)\,[\alpha, x. x])\{\mathsf{id}_\alpha, x/x\}]$$

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The implicit contextual λ-calculus w/ first-class envs. λ_I^{ctx}

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\Gamma, \Psi &::= \alpha \mid \Gamma, x : A \\
\sigma &::= \mathsf{id}_\alpha \mid \sigma, x/M \\
\Sigma &::= \cdot \mid \Sigma; \Gamma
\end{aligned}
$$

$\boxed{\Sigma \vdash M : A}$

$$
\textsc{Box}\ \frac{\Sigma; \Psi \vdash M : A}{\Sigma \vdash [\Psi.M] : [\Psi.A]}
\qquad
\textsc{Unbox}\ \frac{\Sigma \vdash M : [\Psi.A] \qquad \Sigma; \Gamma \vdash \sigma : \Psi}{\Sigma; \Gamma \vdash {\sim}M\{\sigma\} : A}
$$

The implicit contextual λ-calculus w/ first-class envs. λ_I^{ctx}

Running code

- `run` $: (\forall \alpha. [\alpha. A]) \to \forall \alpha. A$

The implicit contextual λ-calculus w/ first-class envs. λ_I^{ctx}

Running code

- `run` $: (\forall \alpha.[\alpha.A]) \rightarrow \forall \alpha.A$
- `subst` $: \forall \alpha.[\alpha, x : A.B] \rightarrow [\alpha.A] \rightarrow [\alpha.B]$

The implicit contextual λ-calculus w/ first-class envs. λ_I^{ctx}

Running code

- $\texttt{run} : (\forall\alpha.[\alpha.A]) \to \forall\alpha.A$
- $\texttt{subst} : \forall\alpha.[\alpha, x : A.B] \to [\alpha.A] \to [\alpha.B]$

Example

$$\lambda f.\Lambda\alpha.[\alpha.\lambda x.\sim(f\,\alpha)\{\mathsf{id}_\alpha, x/x\}]$$
$$: (\forall\alpha.[\alpha, x : A.B]) \to \forall\alpha.[\alpha.A \to B]$$

Going contextual: from λ^{α} to λ_I^{ctx}

There is a transformation from λ^{α} to λ_I^{ctx}, specified as a judgment, and directed by the *derivation*:

$$\boxed{[\![\Xi \vdash^{\bar{\alpha}} E : T]\!] = \Sigma \vdash M : A}$$

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Example

$$[\![\vdash^{\cdot} \lambda f.\, \Lambda\alpha.\, \langle \lambda x.\, {\sim}(f\, \alpha\, \langle x\rangle^\alpha)\rangle^\alpha : (\forall\alpha.\, \langle p\rangle^\alpha \to \langle q\rangle^\alpha) \to \forall\alpha.\, \langle p \to q\rangle^\alpha]\!] =$$
$$\vdash \lambda f.\, \Lambda\alpha.\, [\alpha.\, \lambda x.\, {\sim}(f\, \alpha\, [\alpha, x.\, x])\{\mathsf{id}_\alpha, x/x\}] :$$
$$(\forall\alpha.\, [\alpha.\, p] \to [\alpha.\, q]) \to \forall\alpha.\, [\alpha.\, p \to q]$$

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$$\vdash \Lambda\alpha.\, \lambda f.\, [\alpha.\, \lambda x.\, \sim(f\, [\alpha, x.\, x])\{\mathsf{id}_{\alpha}, x/x\}] :$$
$$\forall\alpha.\, ([\alpha, x : p.\, p] \to [\alpha, x : p.\, q]) \to [\alpha.\, p \to q]$$

Going contextual: from λ^α to λ_I^{ctx}

Definition (Type transformation)

$$\frac{}{[\![p]\!] = p} \qquad \frac{[\![T]\!] = T' \qquad [\![U]\!] = U'}{[\![T \to U]\!] = T' \to U'} \qquad \frac{[\![T]\!] = T'}{[\![\forall \alpha. T]\!] = \forall \alpha. T'}$$

$$\frac{[\![T]\!] = T'}{[\![\langle T \rangle^\alpha]\!] = [\Gamma(\alpha). T']}$$

Γ is *any* environment context; read it as a logic program, not a function.

Going contextual: from λ^α to λ_I^{ctx}

Definition (Derivation transformation)

$$\frac{\llbracket \Xi \vdash^{\bar{\alpha}\alpha} E : T \rrbracket = \Sigma; \Gamma(\alpha) \vdash M : A}{\llbracket \Xi \vdash^{\bar{\alpha}} \langle E \rangle^\alpha : \langle T \rangle^\alpha \rrbracket = \Sigma \vdash [\Gamma(\alpha).M] : [\Gamma(\alpha).A]}\ \textsc{Quote}{\to}\textsc{Box}$$

$$\frac{\llbracket \Xi \vdash^{\bar{\alpha}} E : \langle T \rangle^\alpha \rrbracket = \Sigma \vdash M : [\Psi(\alpha).A]}{\llbracket \Xi \vdash^{\bar{\alpha}\alpha} {\sim}E : T \rrbracket = \Sigma; \Psi(\alpha) \vdash {\sim}M\{\mathsf{id}_{\Psi(\alpha)}\} : A}\ \textsc{Unquote}{\to}\textsc{Unbox}$$

$$\frac{\llbracket \Xi \vdash^{\bar{\alpha}} E : \forall \alpha. T \rrbracket = \Sigma \vdash M : \forall \alpha. A}{\llbracket \Xi \vdash^{\bar{\alpha}} E\, \alpha : T \rrbracket = \Sigma \vdash M\, \Psi(\alpha) : A\{\alpha / \Psi(\alpha)\}}\ \textsc{Inst}{\to}\textsc{Inst}$$

$$\frac{}{\llbracket \Xi \vdash^{\bar{\alpha}} x : T \rrbracket = \llbracket \Xi \rrbracket_{\bar{\alpha}} \vdash x : \llbracket T \rrbracket}\ \textsc{Var}{\to}\textsc{Var}$$

Going contextual: from λ^α to λ_I^{ctx}

Theorem (Correctness)

If $\Xi \vdash^{\bar{\alpha}} E : T$ *and* $[\![\Xi \vdash^{\bar{\alpha}} E : T]\!] = \Sigma \vdash M : A$ *then:*

- $\Sigma \vdash M : A$ *holds*
- $[\![\Xi]\!]_{\bar{\alpha}} = \Sigma$
- $[\![T]\!] = A$.

Theorem (Decidability)

If $\Xi \vdash^{\bar{\alpha}} E : T$ *then* $\exists M$ *s.t.* $[\![\Xi \vdash^{\bar{\alpha}} E : T]\!] = [\![\Xi]\!]_{\bar{\alpha}} \vdash M : [\![T]\!]$.

Going explicit: from λ_I^{ctx} to λ_E^{ctx}

There is a higher-order transformation from λ_I^{ctx} to λ_E^{ctx}, close to *one-pass monadic normal form* transforms (Danvy, 2002):

$$\boxed{[\![\cdot]\!](\cdot) : M_I \to (M_I \to M_E) \to M_E}$$

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Example

$$\begin{aligned}&[\![\lambda f.\,\Lambda\alpha.\,[\alpha.\,\lambda x.\,{\sim}(f\,(\alpha,x:p)\,[\alpha,x.x])\{\mathsf{id}_\alpha,x/x\}\,]]\!] = \\ &\qquad \lambda f.\,\Lambda\alpha.\,\mathsf{let\,box}\;u = f\,(\alpha,x:p)\,[\alpha,x.x]\;\mathsf{in}\;[\alpha.\,\lambda x.\,u\{\mathsf{id}_\alpha,x/x\}]\end{aligned}$$

Going explicit: from λ_I^{ctx} to λ_E^{ctx}

Definition (Term translation)

$$
\begin{aligned}
[\![x]\!](C) &= C(x) \\
[\![\lambda x. M]\!](C) &= [\![M]\!](\texttt{fn}\ M' \to C(\lambda x. M')) \\
[\![M\,N]\!](C) &= [\![M]\!](\texttt{fn}\ M' \to [\![N]\!](\texttt{fn}\ N' \to C(M'\,N'))) \\
[\![[\Gamma. M]]\!](C) &= C([\Gamma. [\![M]\!](\texttt{fn}\ M' \to M')]) \\
[\![\sim M\{\sigma\}]\!](C) &= [\![M]\!](\texttt{fn}\ M' \to [\![\sigma]\!](\texttt{fn}\ \sigma' \to \mathsf{let\,box}\ u = M'\ \mathsf{in}\ C(u\{\sigma'\})))
\end{aligned}
$$

Theorem (Correctness)

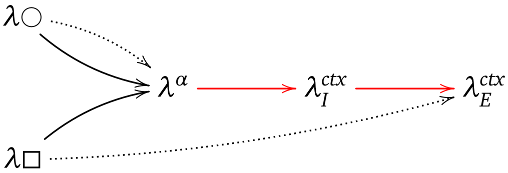
If $\Sigma \vdash M : A$ *then* $[\![\Sigma]\!] \vdash [\![M]\!](\texttt{fn}\ M \to M) : A$

To sum up…

λ^{α} variables annotated with stage

λ_I^{ctx} a stack of environments

λ_E^{ctx} two-zone presentation (validity & truth)



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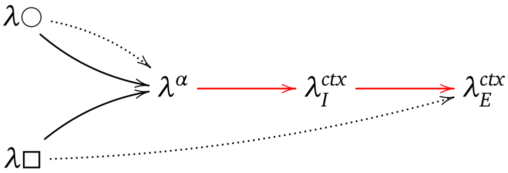

The lessons

- $\lambda^{\alpha} \leq \lambda_E^{ctx}$
- λ_E^{ctx} has a simple operational semantics ($V ::= \lambda x.M \mid [\Psi.M]$)
- there *is* a two-zone presentation of LTL
  (just allow abstraction over environments)
- α range over "approximated" environments

A new look at Normalization

Normalization is evaluation of an annotated program:

$$\lambda x.\,((\lambda y.\,(y+1))\;x) \;:\; \mathsf{nat} \to \mathsf{nat}$$

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Normalization is evaluation of an annotated program:

$$[\alpha.\,\lambda x.\,{\sim}((\lambda y.\,[\alpha, x.\,{\sim}y\{\mathsf{id}_\alpha, x/x\} + 1])[\alpha, x.\,x])\{\mathsf{id}_\alpha, x/x\}] \\ :\ [\alpha.\,\mathsf{nat} \to \mathsf{nat}]$$

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Conjecture (Staging/Binding-time Analysis)

If $M \longrightarrow^* V$ and V a normal form, then there is E s.t. $\texttt{run}\; E \Downarrow V$.

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Conjecture (Staging/Binding-time Analysis)

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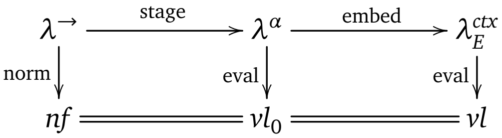
Conjecture (Normalization by staged evaluation)

Staged evaluation "decomposes" normalization:



Conclusion

- contextual types as a logical foundation for staging?
  - strictly subsumes $\lambda\Box$, $\lambda\bigcirc$, $\lambda^\alpha\ldots$
  - in Curry-Howard correspondence with Contextual Logic
- technically: an embedding of environment classifiers into contextual types

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Future works

- relation to NbE?
- pattern-matching on code?
  ex: $\mathsf{case}\ (M : [\alpha.A])\ \mathsf{of}\ [\alpha.M] \to \ldots \mid [\alpha.\#p] \to \ldots$

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Thank you!