Proofs, upside down

A functional correspondence between *natural deduction* and the *sequent calculus*

Matthias Puech

AARHUS UNIVERSITY

State Key Laboratory of Computer Science
Institute of Software, Beijing, December 19, 2013

Logic can explain programs …

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… and programs can explain logic

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Goal of this talk: *understand the relationship between two calculi by means of functional program transformations*

From natural deduction …

$$\textsc{ImpI}\quad \dfrac{\begin{array}{c}[\vdash A]\\ \vdots \\ \vdash B\end{array}}{\vdash A \supset B} \qquad\qquad \textsc{ImpE}\quad \dfrac{\vdash A \supset B \qquad \vdash A}{\vdash B}$$

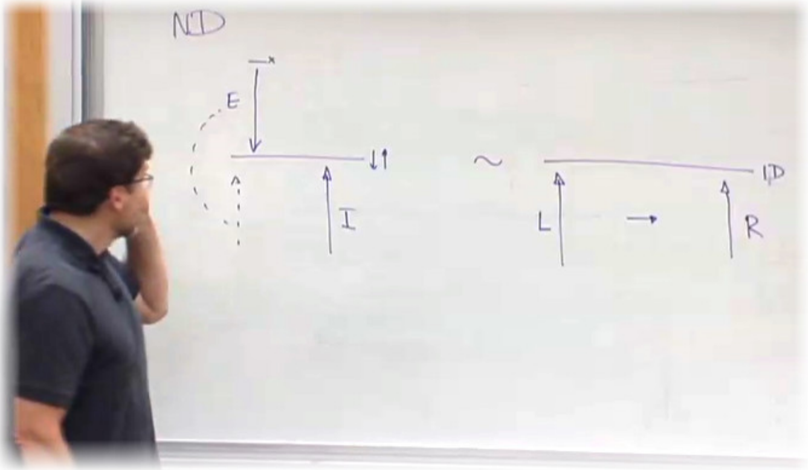
- “natural” reasoning steps
- inferences change the goal, hypotheses and “hanging”
- bidirectional reading, difficult proof search

… to the sequent calculus

$$\textsc{ImpR}\ \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \qquad \textsc{ImpL}\ \frac{\Gamma \longrightarrow A \qquad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}$$

- “fine-grained” reasoning steps
- left inferences change hypotheses
- bottom-up reading, easy proof search

An intuition



Natural deductions are "reversed" sequent calculus proofs

An intuition

Example (The Barbara syllogism)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\vdash (q \supset r)] \qquad \dfrac{[\vdash (p \supset q)] \qquad [\vdash p]}{\vdash q}\;\textsc{ImpE}}{\vdash r}\;\textsc{ImpE}}{\vdash p \supset r}\;\textsc{ImpI}}{\vdash (q \supset r) \supset p \supset r}\;\textsc{ImpI}}{\vdash (p \supset q) \supset (q \supset r) \supset p \supset r}\;\textsc{ImpI}
$$

An intuition

Example (The Barbara syllogism)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \longrightarrow p}\,\textsc{Id} \qquad \dfrac{\dfrac{}{q \longrightarrow q}\,\textsc{Id} \qquad \dfrac{}{r \longrightarrow r}\,\textsc{Id}}{q \supset r, q \longrightarrow r}\,\textsc{ImpL}}{p \supset q, q \supset r, p \longrightarrow r}\,\textsc{ImpL}}{p \supset q, q \supset r \longrightarrow p \supset r}\,\textsc{ImpR}}{p \supset q \longrightarrow (q \supset r) \supset p \supset r}\,\textsc{ImpR}}{\longrightarrow (p \supset q) \supset (q \supset r) \supset p \supset r}\,\textsc{ImpR}$$

An intuition

Problem

How to make this intuition formal?

- how to define “reversal” generically?
- from N.D., how to *derive* S.C.?

and now, for something completely different…

Accumulator-passing style

A well-known programmer trick to save stack space

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- a function in direct style:

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let rec tower1 = function
  | [] → 1
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- the same in accumulator-passing style:

```
let rec tower2 acc = function
  | [] → acc
  | x :: xs → tower2 (x ** acc) xs
```

Accumulator-passing style

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let rec tower1 = function
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- the same in accumulator-passing style:

```
let rec tower2 acc = function
  | [] → acc
  | x :: xs → tower2 (x ** acc) xs

(* don't forget to reverse the input list *)
let tower xs = tower2 1 (List.rev xs)
```

In this talk

$$\frac{\text{sequent calculus}}{\text{natural deduction}} = \frac{\texttt{tower2}}{\texttt{tower1}}$$

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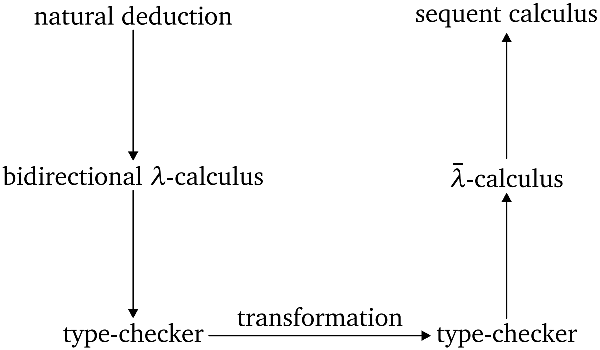
The message

- S.C. is an accumulator-passing N.D.
- there is a systematic, off-the-shelf transformation from N.D.-style systems to S.C.-style systems
- it is modular, i.e., it applies to variants of N.D./S.C.
- a programmatic explanation of a proof-theoretical artifact

In this talk

The medium

Go through term assignments and reason on the type checker:



Outline

Outline

The transformation

Some extensions

Starting point: the Bidirectional λ-calculus

a.k.a. intercalations, normal forms+annotation [Pierce and Turner, 2000]

$\boxed{\vdash A \downarrow}$ Use

$$\frac{\vdash A \supset B \downarrow \qquad \vdash A \uparrow}{\vdash B \downarrow}\ \textsc{App} \qquad\qquad \frac{\vdash A \uparrow}{\vdash A \downarrow}\ \textsc{Annot}$$

$\boxed{\vdash A \uparrow}$ Verification

$$\frac{\begin{array}{c}[\vdash A \downarrow] \\ \vdots \\ \vdash B \uparrow\end{array}}{\vdash A \supset B \uparrow}\ \textsc{Lam} \qquad\qquad \frac{\vdash A \downarrow}{\vdash A \uparrow}\ \textsc{Atom}$$

Starting point: the Bidirectional λ-calculus

a.k.a. intercalations, normal forms+annotation [Pierce and Turner, 2000]

$$A ::= \mathsf{p} \mid A \supset A \qquad \text{Types}$$
$$M ::= \lambda x.M \mid R \qquad \text{Terms}$$
$$R ::= R\,M \mid x \mid (M : A) \qquad \text{Atoms}$$

$\boxed{\Gamma \vdash R \Rightarrow A}$ Inference

$$\text{Var}\ \frac{x : A \in \Gamma}{\Gamma \vdash x \Rightarrow A} \qquad \text{App}\ \frac{\Gamma \vdash R \Rightarrow A \supset B \qquad \Gamma \vdash M \Leftarrow A}{\Gamma \vdash R\,M \Rightarrow B} \qquad \text{Annot}\ \frac{\Gamma \vdash M \Leftarrow A}{\Gamma \vdash (M : A) \Rightarrow A}$$

$\boxed{\Gamma \vdash M \Leftarrow A}$ Checking

$$\text{Lam}\ \frac{\Gamma, x : A \vdash M \Leftarrow B}{\Gamma \vdash \lambda x.M \Leftarrow A \supset B} \qquad \text{Atom}\ \frac{\Gamma \vdash R \Rightarrow C}{\Gamma \vdash R \Leftarrow C}$$

Starting point: the Bidirectional λ-calculus

```
type a = Base | Imp of a × a
type m = Lam of string × m | Atom of r
and r = App of r × m | Var of string | Annot of m × a

let rec check env c : m → unit =
 let rec infer : r → a = fun r → match r with
 | Var x → List.assoc x env
 | Annot (m, a) → check env a m; a
 | App (r, m) → let Imp (a, b) = infer r in check env a m; b
 in fun m → match m, c with
  | Lam (x, m), Imp (a, b) → check ((x, a) :: env) b m
  | Atom r, _ → match infer r with c’ when c=c’ → ()
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Remarks

- inference in constant environment → infer λ-dropped

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```

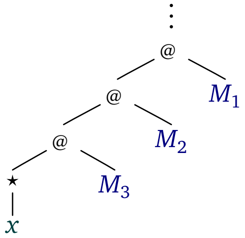
Remarks

- inference in constant environment → `infer` λ-dropped
- `infer` is head-recursive

Inefficiency: no tail recursion

```
(* ... *)
 let rec infer : r → a = fun r → match r with
 | Var x → List.assoc x env
 | Annot (m, a) → check env a m; a
 | App (r, m) → let Imp (a, b) = infer r in check env a m; b
(* ... *)
```

Example



Solution: reverse atomic terms

```
(* ... *)
 let rec infer : r → a = fun r → match r with
  | Var x → List.assoc x env
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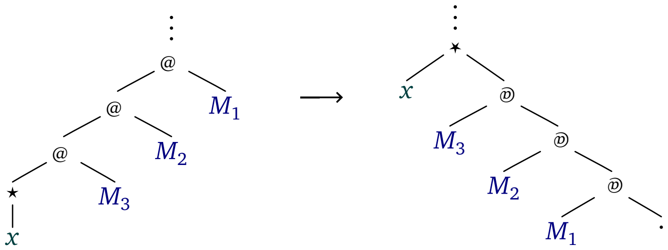

The transformation

An application of Danvy and Nielsen [2001]'s framework:

- (partial) *CPS transformation*
- (lightweight) *defunctionalization*
- *reforestation* (= deforestation^{-1})

Turns *direct style* into *accumulator-passing style*

Step 1. CPS transformation of infer (call-by-value)

```
let rec check env c : m → unit =
  let rec infer : r → a = fun r → match r with
    | Var x → List.assoc x env
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  in fun m → match m, c with
    | Lam (x, m), Imp (a, b) → check ((x, a) :: env) b m
    | Atom r, _ → match infer r with c' when c=c' → ()
```

Step 1. CPS transformation of `infer` (call-by-value)

```
type k = a → unit
let rec check env c : m → unit =
 let rec infer : r → k → unit = fun r k → match r with
   | Var x → k (List.assoc x env)
   | Annot (m, a) → check env a m; k a
   | App (r, m) → infer r (fun (Imp (a, b)) → check env a m; k b)
 in fun m → match m, c with
   | Lam (x, m), Imp (a, b) → check ((x, a) :: env) b m
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Step 2. (lightweight) Defunctionalization

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    | Var x → k (List.assoc x env)
    | Annot (m, a) → check env a m; k a                (* KCons *)
    | App (r, m) → infer r (fun (Imp (a, b)) → check env a m; k b)
  in fun m → match m, c with
    | Lam (x, m), Imp (a, b) → check ((x, a) :: env) b m
    | Atom r, _ → infer r (function c' when c=c' → ()) (* KNil *)
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    | Atom r, _ → infer r KNil
```

Step 2. (lightweight) Defunctionalization

```
type k = KNil | KCons of m × k
let rec check env c : m → unit =
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Step 2. (lightweight) Defunctionalization

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type k = KNil | KCons of m × k
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    | KCons (m, k), Imp (a, b) → check env a m; apply k b in
  let rec infer : r → k → unit = fun r k → match r with
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Step 2. (lightweight) Defunctionalization

```
type k = KNil | KCons of m × k
let rec check env c : m → unit =
  let rec cont : k → a → unit = fun k a → match k, a with
    | KNil, c’ when c=c’ → ()
    | KCons (m, k), Imp (a, b) → check env a m; cont k b in
  let rec rev_atom : r → k → unit = fun r k → match r with
    | Var x → cont k (List.assoc x env)
    | Annot (m, a) → check env a m; cont k a
    | App (r, m) → rev_atom r (KCons (m, k))
  in fun m → match m, c with
    | Lam (x, m), Imp (a, b) → check ((x, a) :: env) b m
    | Atom r, _ → rev_atom r KNil
```

Step 3. Reforestation

Goal

Introduce intermediate data structure of *reversed term* V
to decouple *reversal* from *checking*:

$$\texttt{check} \circ \texttt{rev\_atom} \circ \texttt{cont}$$
$$\downarrow$$
$$\texttt{rev} \circ \texttt{check} \circ \texttt{cont}$$

Step 3. Reforestation

```
(* intermediate data structure *)
type v = VLam of string × v | VHead of h
and h =
  | HVar of string × k
  | HAnnot of v × a × k
and k = KNil | KCons of v × k
```

Step 3. Reforestation

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(* intermediate data structure *)
type v = VLam of string × v | VHead of h
and h =
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and k = KNil | KCons of v × k

(* term reversal *)
let rec rev : m → v = fun m → match m with
  | Lam (x, m) → VLam (x, rev m)
  | Atom r → VHead (rev_atom r KNil)
and rev_atom : r → k → h = fun r k → match r with
  | Var x → HVar (x, k)
  | Annot (m, a) → HAnnot (rev m, a, k)
  | App (r, m) → rev_atom r (KCons (rev m, k))
```

Step 3. Reforestation

```
(* reversed term checking *)
let rec check env c : v → unit =
  let rec cont : k → a → unit = fun k a → match k, a with
    | KNil, c' when c=c' → ()
    | KCons (m, k), Imp (a, b) → check env a m; cont k b in
  let head h = match h with
    | HVar (x, k) → cont k (List.assoc x env)
    | HAnnot (m, a, k) → check env a m; cont k a in
  fun v → match v, c with
    | VLam (x, m), Imp (a, b) → check ((x, a) :: env) b m
    | VHead h, _ → head h

(* main function *)
let check env c m = check env c (rev m)
```

End result: the $\bar{\lambda}$-calculus

a.k.a. *spine calculus*, or LJT, or *n*-ary application [Herbelin, 1994]

$V ::= \lambda x.V \mid H$	Values
$H ::= x(S) \mid (V:A)(S)$	Heads
$S ::= \cdot \mid V,S$	Spines

$\boxed{\Gamma \mid A \longrightarrow S : C}$ Focused left rules

$$\textsc{SApp}\ \frac{\Gamma \longrightarrow V : A \qquad \Gamma \mid B \longrightarrow S : C}{\Gamma \mid A \supset B \longrightarrow V,S : C}$$

$$\textsc{SAtom}\ \frac{}{\Gamma \mid C \longrightarrow \cdot : C}$$

$\boxed{\Gamma \longrightarrow V : A}$ Right rules

$$\textsc{VLam}\ \frac{\Gamma, x:A \longrightarrow V : B}{\Gamma \longrightarrow \lambda x.M : A \supset B}$$

$$\textsc{HVar}\ \frac{x:A \in \Gamma \qquad \Gamma \mid A \longrightarrow S : C}{\Gamma \longrightarrow x(S) : C}$$

$$\textsc{HAnnot}\ \frac{\Gamma \longrightarrow V : A \qquad \Gamma \mid A \longrightarrow S : C}{\Gamma \longrightarrow (V:A)(S) : C}$$

End result: the $\bar{\lambda}$-calculus

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\begin{array}{rcll}
V & ::= & \lambda x.\, V \mid H & \text{Values} \\
H & ::= & x\,(S) \mid (V : A)\,(S) & \text{Heads} \\
S & ::= & \cdot \mid V, S & \text{Spines}
\end{array}
$$

$\boxed{\Gamma \mid A \longrightarrow C}$ Focused left rules

$$
\frac{\Gamma \longrightarrow A \qquad \Gamma \mid B \longrightarrow C}{\Gamma \mid A \supset B \longrightarrow C}\ \textsc{SApp}
\qquad
\frac{}{\Gamma \mid C \longrightarrow C}\ \textsc{SAtom}
$$

$\boxed{\Gamma \longrightarrow A}$ Right rules

$$
\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \textsc{VLam}
\qquad
\frac{A \in \Gamma \qquad \Gamma \mid A \longrightarrow C}{\Gamma \longrightarrow C}\ \textsc{HVar}
\qquad
\frac{\Gamma \longrightarrow A \qquad \Gamma \mid A \longrightarrow C}{\Gamma \longrightarrow C}\ \textsc{HAnnot}
$$

End result: the $\bar{\lambda}$-calculus

Example

In the bidirectional λ-calculus:

$$\lambda x.(((x\, M_1)\, M_2)\, M_3)$$

In the $\bar{\lambda}$-calculus:

$$\lambda x.x\,(M_1, M_2, M_3)$$

End result: the $\bar{\lambda}$-calculus

Example

In the bidirectional λ-calculus:

$$\lambda x. (((x\, M_1)\, M_2)\, M_3)$$

In the $\bar{\lambda}$-calculus:

$$\lambda x. x\, (M_1, M_2, M_3)$$

Theorem

Initial.`check env a m` $=$ *()* *iff* *Final*.`check env a m` $=$ *()*

Proof.

By composition of the soundness of the transformations □

End result: the $\bar{\lambda}$-calculus

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Theorem

$\Gamma \vdash M \Leftarrow A$ *iff* $\Gamma \longrightarrow (\texttt{rev}\,M) : A$

Proof.

By composition of the soundness of the transformations □

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$\Gamma \vdash A$ *iff* $\Gamma \longrightarrow A$

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$\Gamma \vdash A$ *iff* $\Gamma \longrightarrow A$

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By composition of the soundness of the transformations □

Remark

we derived the rules of LJT

Outline

Extension 1. Full propositional intuitionistic N.D.

It scales to full NJ: $A \wedge B$ and $A \vee B$ [Herbelin, 1995]:

$$\dfrac{\vdash A \vee B \downarrow \qquad \begin{array}{c}[\vdash A \downarrow] \\ \vdots \\ \vdash C \uparrow\end{array} \qquad \begin{array}{c}[\vdash B \downarrow] \\ \vdots \\ \vdash C \uparrow\end{array}}{\vdash C \uparrow}\ \textsc{DisjE}$$

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Term assignment:

$$M ::= \lambda x.\, M \;\Big|\; \langle M, M \rangle \;\Big|\; \mathbf{inl}(M) \;\Big|\; \mathbf{inr}(M) \;\Big|\; \mathbf{case}\ R\ \mathbf{of}\ \langle x.\, M \mid x.\, M \rangle \;\Big|\; R$$

$$R ::= x \;\Big|\; R\, M \;\Big|\; \pi_1(R) \;\Big|\; \pi_2(R) \;\Big|\; (M : A)$$

Extension 1. Full propositional intuitionistic N.D.

It scales to full NJ: $A \wedge B$ and $A \vee B$ [Herbelin, 1995]:

$$\dfrac{\vdash A \vee B \downarrow \qquad \begin{array}{c}[\vdash A \downarrow]\\ \vdots \\ \vdash C \uparrow\end{array} \qquad \begin{array}{c}[\vdash B \downarrow]\\ \vdots \\ \vdash C \uparrow\end{array}}{\vdash C \uparrow}\ \textsc{DisjE}$$

Term assignment:

$$M ::= \lambda x.M \mid \langle M,M\rangle \mid \mathbf{inl}(M) \mid \mathbf{inr}(M) \mid \mathbf{case}\ R\ \mathbf{of}\ \langle x.M \mid x.M\rangle \mid R$$

$$R ::= x \mid R\,M \mid \pi_1(R) \mid \pi_2(R) \mid (M : A)$$

Reversed terms:

$$V ::= \lambda x.V \mid \langle V,V\rangle \mid \mathbf{inl}(V) \mid \mathbf{inr}(V) \mid x\,(S) \mid (M : A)\,(S)$$

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$$\textsc{DisjL}\ \dfrac{\Gamma, x : A \longrightarrow V_1 : C \qquad \Gamma, y : B \longrightarrow V_2 : C}{\Gamma \mid A \vee B \longrightarrow \mathbf{case}\langle x.V_1 \mid y.V_2\rangle : C}$$

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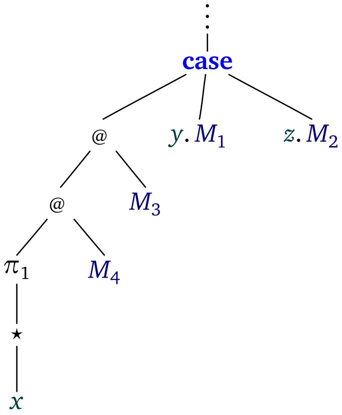
Example



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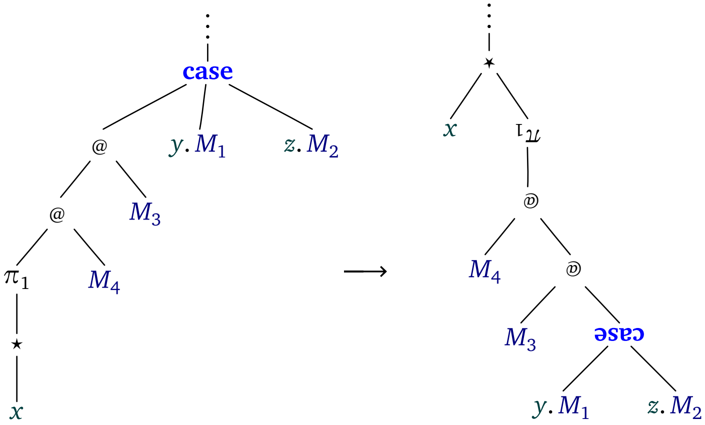

Extension 2. Multiplicative connectives

We can define conjunction multiplicatively [Girard et al., 1989]:

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Extension 3. Unfocused sequent calculus

Let us add a *cut* rule to N.D. [Espírito Santo, 2007]:

$$\dfrac{\vdash A\downarrow \qquad \begin{array}{c}[\vdash A\downarrow]\\ \vdots \\ \vdash B\uparrow\end{array}}{\vdash B\uparrow}\ \textsc{Cut}$$

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Extension 4. A modal logic of necessity

We can introduce a *necessity operator*: [Pfenning and Davies, 2001]

$$\textsc{BoxI}\ \frac{\Delta; \cdot \vdash A}{\Delta; \Gamma \vdash \Box A}$$

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Take the term assignment of LJQ (LJT in call by value).

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Conclusion

- a systematic derivation of S.C.-style calculi from N.D.-style calculi, using “algebraic” CPS $\circ$ reforestation
- N.D. terms + checker $\longrightarrow$ S.C. terms + reversal + checker
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Thank you!

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