CERTIFICATES FOR INCREMENTAL TYPE CHECKING

Thesis defense

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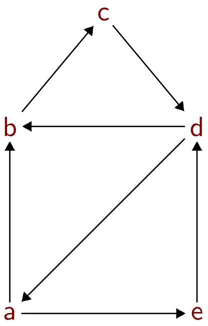
Outline

Quiz

Is there a cycle of size 2^N in this graph?



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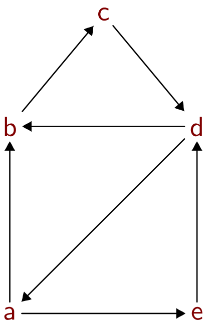

val `cycle` : `graph` → `bool`

The algorithm and its specification

Specification

$$\frac{A \to B}{A \leadsto_0 B}\ \textsc{Edge} \qquad \frac{A \leadsto_N B \qquad B \leadsto_N C}{A \leadsto_{\mathsf{s}(N)} C}\ \textsc{Trans}$$

Quiz

Given $\mathsf{a} \to \mathsf{b}$, $\mathsf{b} \to \mathsf{c}$, $\mathsf{c} \to \mathsf{d}$, $\mathsf{d} \to \mathsf{a}$, $\mathsf{d} \to \mathsf{b}$, $\mathsf{e} \to \mathsf{d}$, $\mathsf{a} \to \mathsf{e}$, are there A and N such that $A \leadsto_N A$ holds?

The algorithm and its specification

Specification

$$\textsc{Edge}\ \frac{A \to B}{A \rightsquigarrow_0 B} \qquad \textsc{Trans}\ \frac{A \rightsquigarrow_N B \qquad B \rightsquigarrow_N C}{A \rightsquigarrow_{\mathsf{s}(N)} C}$$

Quiz

Given $\mathsf{a} \to \mathsf{b}$, $\mathsf{b} \to \mathsf{c}$, $\mathsf{c} \to \mathsf{d}$, $\mathsf{d} \to \mathsf{a}$, $\mathsf{d} \to \mathsf{b}$, $\mathsf{e} \to \mathsf{d}$, $\mathsf{a} \to \mathsf{e}$, are there A and N such that $A \rightsquigarrow_N A$ holds? There are ($A = \mathsf{a}$, $N = 2$):

$$\dfrac{\dfrac{\dfrac{\overline{\mathsf{a} \to \mathsf{b}}}{\mathsf{a} \rightsquigarrow_0 \mathsf{b}}\ \textsc{Edge} \qquad \dfrac{\overline{\mathsf{b} \to \mathsf{c}}}{\mathsf{b} \rightsquigarrow_0 \mathsf{c}}\ \textsc{Edge}}{\mathsf{a} \rightsquigarrow_1 \mathsf{c}}\ \textsc{Trans} \qquad \dfrac{\dfrac{\overline{\mathsf{c} \to \mathsf{d}}}{\mathsf{c} \rightsquigarrow_0 \mathsf{d}}\ \textsc{Edge} \qquad \dfrac{\overline{\mathsf{d} \to \mathsf{a}}}{\mathsf{d} \rightsquigarrow_0 \mathsf{a}}\ \textsc{Edge}}{\mathsf{c} \rightsquigarrow_1 \mathsf{a}}\ \textsc{Trans}}{\mathsf{a} \rightsquigarrow_2 \mathsf{a}}\ \textsc{Trans}$$

Quiz

Is this true?

$$(p \supset q) \supset (p \vee r) \supset (q \vee r)$$

Quiz

Is this true?

$$(p \supset q) \supset (p \vee r) \supset (q \vee r)$$

val `prove` : `prop` → `bool`

The statement and its proof

Specification

$$\frac{\vdash A \supset B \qquad \vdash A}{\vdash B}\ \textsc{ImpE} \qquad \frac{\vdash A}{\vdash A \lor B}\ \textsc{DisjI1} \qquad \frac{\vdash B}{\vdash A \lor B}\ \textsc{DisjI2}$$

$$\frac{\begin{matrix}[\vdash A]\\ \vdots \\ \vdash B\end{matrix}}{\vdash A \supset B}\ \textsc{ImpI} \qquad \frac{\vdash A \lor B \qquad \begin{matrix}[\vdash A]\\ \vdots \\ \vdash C\end{matrix} \qquad \begin{matrix}[\vdash B]\\ \vdots \\ \vdash C\end{matrix}}{\vdash C}\ \textsc{DisjE}$$

Quiz

Does $\vdash (\mathsf{p} \supset \mathsf{q}) \supset (\mathsf{p} \lor \mathsf{r}) \supset (\mathsf{q} \lor \mathsf{r})$ hold?

The statement and its proof

Specification

$$\textsc{ImpE}\ \dfrac{\vdash A \supset B \qquad \vdash A}{\vdash B} \qquad\qquad \textsc{DisjI1}\ \dfrac{\vdash A}{\vdash A \vee B} \qquad\qquad \textsc{DisjI2}\ \dfrac{\vdash B}{\vdash A \vee B}$$

$$\dfrac{\begin{array}{c}[\vdash A]\\ \vdots \\ \vdash B\end{array}}{\vdash A \supset B}\ \textsc{ImpI} \qquad\qquad \dfrac{\vdash A \vee B \qquad \begin{array}{c}[\vdash A]\\ \vdots \\ \vdash C\end{array} \qquad \begin{array}{c}[\vdash B]\\ \vdots \\ \vdash C\end{array}}{\vdash C}\ \textsc{DisjE}$$

Quiz

Does $\vdash (p \supset q) \supset (p \vee r) \supset (q \vee r)$ hold? It does:

$$\dfrac{\dfrac{\dfrac{[\vdash (p \vee r)] \qquad \dfrac{\dfrac{[\vdash (p \supset q)] \qquad [\vdash p]}{\vdash q}\ \textsc{ImpE}}{\vdash q \vee r}\ \textsc{DisjI1} \qquad \dfrac{[\vdash r]}{\vdash q \vee r}\ \textsc{DisjI2}}{\vdash q \vee r}\ \textsc{DisjE}}{\vdash (p \vee r) \supset q \vee r}\ \textsc{ImpI}}{\vdash (p \supset q) \supset (p \vee r) \supset q \vee r}\ \textsc{ImpI}$$

Declarative specifications, performative algorithms

A program can be arbitrarily more complex than its specification.
How can I be sure that a program respects its specification?

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How can I be sure that a program respects its specification?

Dynamically

```
let cycle : graph → path option = ...
verify (cycle [A, B; A, C; B, C])
```

Declarative specifications, performative algorithms

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How can I be sure that a program respects its specification?

Dynamically

let `cycle` : `graph` → `path option` = ...
`verify` (`cycle` [A, B; A, C; B, C])

Statically

let `cycle` : `graph` → `bool` = ...
Theorem for all G, `cycle` G = **true** *iff* G has a 2^N-cycle

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How can I be sure that a program respects its specification?

Dynamically

```
let cycle : graph → path option = ...
verify (cycle [A, B; A, C; B, C])
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Statically

```
let cycle : graph → bool = ...
```

Theorem for all G, `cycle` G = **true** *iff* G has a 2^N-cycle

In both cases, external tools are required

Thesis

We can integrate a program and its specification
by developping programming tools,
and relying on proof theory

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We can integrate a program and its specification
by developping programming tools,
and relying on proof theory

Argued by two main contributions:

1. programming with proof certificates
2. incremental type checking

Outline

Proof certificates

Witnesses of correctness of a computation,
independently verifiable by a small trusted program

- a specification of f, e.g. for all i, if $f(i) = o$ then $\mathsf{P}(i, o)$
- an untrusted oracle computing $f(i) = \langle o, \pi \rangle$
- a trusted *kernel* deciding if π is a proof of $\mathsf{P}(i, o)$

Examples

Proof-carrying code [Necula, 1997], *theorem proving* [Asperti and Tassi, 2007], *safe type inference* [Vytiniotis, 2008]

Proof certificates

Question

How to write programs manipulating proofs?

It is notably hard to write *correct* certificate-issuing programs with a *general-purpose* programming language:

1. how to represent proofs?
2. how to manipulate hypotheses? (binders)
3. how to compute on hypothetical proofs? (free variables)

1. A data structure for proofs?

Question

How to *represent* proofs?

Example

```
type vertex = string
type edge = vertex × vertex
type path =
  | Edge of edge
  | Trans of path × path

let check l : path → bool = …
```

1. A data structure for proofs?

Question

How to *represent* proofs?

Example

```
type prop = Atom of string | Imp of prop × prop
type pf =
  | DisjI1 of prop × pf | DisjI2 of prop × pf
  | ImpE of pf  × pf
  | ImpI of string × pf
  | DisjE of pf × string × pf × string × pf
let check l : pf → bool = ...
```

1. LF: a universal notation for hypothetical proofs

- many formal systems and logics
- a common *hypothetical reasoning* core
  e.g. logics with dischargeable hypotheses, programming languages with variable binders

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Proposition

Use LF [Harper et al., 1993] as the *values* manipulated

LF is a universal *representation language*, for formal systems featuring hypothetical reasoning, like HTML for structured documents

- systems (*resp.* derivations) encoded into *signatures* (*resp. objects*)
- dependently-typed λ-calculus ($\lambda\Pi$)
- *higher-order abstract syntax*

1. LF: a universal notation for hypothetical proofs

Example (Encoding natural deductions)

$$\left(\begin{array}{c} \dfrac{\begin{array}{c}[\vdash A]\\ \vdots \\ \vdash B\end{array}}{\vdash A \supset B}\,\textsc{ImpI} \\[2em] \dfrac{\vdash A \supset B \qquad \vdash A}{\vdash B}\,\textsc{ImpE} \end{array} \right)^{*} = \left(\begin{array}{l} \mathsf{prop} : *. \\ \mathsf{p} : \mathsf{prop}.\ \mathsf{q} : \mathsf{prop}. \\ \mathsf{imp} : \mathsf{prop} \to \mathsf{prop} \to \mathsf{prop}. \\ \mathsf{pf} : \mathsf{prop} \to *. \\ \mathsf{ImpI} : \Pi A B : \mathsf{prop}. \\ \quad (\mathsf{pf}\ A \to \mathsf{pf}\ B) \to \mathsf{pf}\ (\mathsf{imp}\ A\ B). \\ \mathsf{ImpE} : \Pi A B : \mathsf{prop}. \\ \quad \mathsf{pf}\ (\mathsf{imp}\ A\ B) \to \mathsf{pf}\ A \to \mathsf{pf}\ B. \end{array} \right)$$

$$\left(\dfrac{\dfrac{[\vdash \mathsf{p}]}{\vdash \mathsf{q} \supset \mathsf{p}}\,\textsc{ImpI}}{\vdash \mathsf{p} \supset \mathsf{q} \supset \mathsf{p}}\,\textsc{ImpI} \right)^{*} = \left(\begin{array}{l} \mathsf{ImpI}\ \mathsf{p}\ (\mathsf{imp}\ \mathsf{q}\ \mathsf{p}) \\ \quad (\lambda x.\, \mathsf{ImpI}\ \mathsf{q}\ \mathsf{p}\ (\lambda y.x)) \end{array} \right)$$

Computing LF objects?

Question

How to write programs which *values* are LF *objects*?

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Related work

Twelf [Pfenning and Schürmann, 1999], Beluga [Pientka and Dunfield, 2010], Delphin [Poswolsky and Schürmann, 2008]

Computing LF objects?

Question

How to write programs which *values* are LF *objects*?

Related work

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Proposition

An OCaml library to ease programming with proof certificates:

- ✓ general purpose functional programming language
- ✓ large corpus of libraries
- ✗ only simply-typed (ADT)

OCaml + LF = Gasp

http://www.cs.unibo.it/~puech/

Gasp: an OCaml library to manipulate LF objects

An implementation of LF…

- a type of LF objects `obj`
- a type of signatures `sign`
- a concrete syntax with quotations and anti-quotations
  e.g. $(\text{«}^{sign}$ prop : $*$. imp : prop $\rightarrow$ prop $\rightarrow$ prop. » : `sign`)
  e.g. **fun** (`x`:`obj`) $\rightarrow$ (« imp "`x`" "`x`" » : `obj`)

Gasp: an OCaml library to manipulate LF objects

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- a type of LF objects `obj`
- a type of signatures `sign`
- a concrete syntax with quotations and anti-quotations
  e.g. («^{sign} prop : ∗. imp : prop → prop → prop. » : `sign`)
  e.g. **fun** (`x`:`obj`) → (« imp "`x`" "`x`" » : `obj`)

…where signatures can declare *functions symbols f*

- their code is untyped OCaml code (type `obj`)
  can use e.g. pattern-matching, exceptions, partiality…
- their LF types act as a specifications
  dynamically checked at run time

Gasp: an OCaml library to manipulate LF objects

Example

```
# let s = s ++ «sign
  tryProveIdentity : Πx : prop. pf x = "fun x → match x with
      | « imp "y" "z" » → « Impl "y" "z" (λx.x) »".
» ;;
s : sign = «sign ... »
```

Gasp: an OCaml library to manipulate LF objects

Example

```
# let s = s ++ «sign
  tryProveIdentity : Πx : prop. pf x = "fun x → match x with
      | « imp "y" "z" » → « Impl "y" "z" (λx.x) »".
» ;;
s : sign = «sign ... »

# eval s « tryProveIdentity (imp p p) » ;;
- : obj = « Impl p p (λx.x) »
```

Gasp: an OCaml library to manipulate LF objects

Example

```
# let s = s ++ «sign
  tryProveIdentity : Πx : prop. pf x = "fun x → match x with
      | « imp "y" "z" » → « Impl "y" "z" (λx.x) »".
» ;;
s : sign = «sign ... »

# eval s « tryProveIdentity (imp p p) » ;;
- : obj = « Impl p p (λx.x) »

# eval s « tryProveIdentity (imp p (imp q p)) » ;;
Exception: Type_error (« λx.x », « pf p → pf (imp q p) »)
```

2. Manipulating values with binders?

Consider the signature of the λ-calculus

```
# let s = «sign tm : ∗.
   app : tm → tm → tm.   lam : (tm → tm) → tm. » ;;
```

and function

```
# let eta_expand m = « lam λx. (app "m" x) » ;;
```

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let eta_expand m = « lam λx. (app "m" x) » ;;

What is the value of this expression?

« lam λx. "eta_expand « x »" »

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Consider the signature of the λ-calculus

$\#$ **let** s = «^{sign} tm : $*$.
  app : tm $\rightarrow$ tm $\rightarrow$ tm.   lam : (tm $\rightarrow$ tm) $\rightarrow$ tm. » ;;

and function

$\#$ **let** eta_expand m = « lam λx. (app "m" x) » ;;

What is the value of this expression?

« lam λx. (lam λx. app x x) »

$\rightsquigarrow$ *Variable capture during substitution*

Related work
FreshML [Shinwell et al., 2003], Cαml [Pottier, 2007], [McBride and McKinna, 2004], [Pouillard and Pottier, 2010]

2. Safe renaming & substitution in Gasp

let s = «^{sign}
tm : ∗. app : tm → tm → tm. lam : (tm → tm) → tm.
etaExpand : tm → tm = "**fun** m → « lam (λ*x*. app "m" *x*) »". » ;;

2. Safe renaming & substitution in Gasp

```
# let s = «sign
  tm : ∗.   app : tm → tm → tm.   lam : (tm → tm) → tm.
  etaExpand : tm → tm = "fun m → « lam (λx. app "m" x) »". » ;;

# eval s « lam (λx. etaExpand x) » ;;
```

2. Safe renaming & substitution in Gasp

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# let s = «sign
  tm : *.   app : tm → tm → tm.   lam : (tm → tm) → tm.
  etaExpand : tm → tm = "fun m → « lam (λx. app "m" x) »". » ;;

# eval s « lam (λx. etaExpand x) » ;;
- : obj = « lam (λx. lam (λx'. app x x')) »
```

2. Safe renaming & substitution in Gasp

```
# let s = «sign
  tm : ∗.   app : tm → tm → tm.   lam : (tm → tm) → tm.
  etaExpand : tm → tm = "fun m → « lam (λx. app "m" x) »". » ;;

# eval s « lam (λx. etaExpand x) » ;;
- : obj = « lam (λx. lam (λx'. app x x')) »

# let s = s ++ «sign
  vl : ∗.   vlam : (tm → tm) → vl.
  interpret : tm → vl = "fun m → match m with
    | « lam "m" » → « vlam "m" »
    | « app "m" "n" » →
        let « vlam "p" » = « interpret "m" » in
        « interpret ("p" "n") »". » ;;
```

2. Safe renaming & substitution in Gasp

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  tm : ∗.   app : tm → tm → tm.   lam : (tm → tm) → tm.
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    | « app "m" "n" » →
        let « vlam "p" » = « interpret "m" » in
        « interpret ("p" "n") »". » ;;

# eval s « interpret (app (lam (λx.x)) (lam (λx.x))) » ;;
- : obj = « vlam (λx.x) »
```

2. Safe renaming & substitution in Gasp

Contribution

✓ Substitution and α-renaming are handled by the tool, thanks to an original *locally named* representation of concrete LF objects:

- free variables are *numbered*
  protected against capture
- bound variables are *named*
  written by the user

Example

«lam ($\lambda x.$ *etaExpand* x)»

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Example

«lam *(λx. etaExpand #1)*»

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«lam $(\lambda x.$ lam $(\lambda x.$ app $\#2\ x))$»

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Example

«lam $(\lambda x.$ lam $(\lambda x'.$ app $x\ x'))$»

3. Computing on objects with free variables?

Consider the *size* function $|\cdot|$ defined recursively on λ-terms:

$$
\begin{aligned}
|x| &= 0 \\
|\lambda x.\, M| &= |M| + 1 \\
|M\, N| &= |M| + |N| + 1
\end{aligned}
$$

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Consider the *size* function $|\cdot|$ defined recursively on λ-terms:

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$$

Can we code it as follows?

```
size : tm → nat = "fun m → match m with
  | « x » → « o »
  | « app "m" "n" » → « s (plus (size "m") (size "n")) »
  | « lam "f" » → « s (size "f") » ".
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```

⇝ *Ill-typed:* f *has type* tm → tm

Related work

[Miller and Palamidessi, 1999], Contextual Modal Type Theory [Nanevski et al., 2008], Abella [Gacek et al., 2011]

3. *Environment-free* computation in Gasp

Proposition

Consider *size* only called on closed objects (no variable case), introduce *function inverse* $size^0$: nat → tm feeding *output* back to *input*

```
size : tm → nat = "fun m → match m with
  | « app "m" "n" » → « s (plus (size "m") (size "n")) »
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```

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We add *contraction* to the reduction

$size\ (size^0\ M) = M$

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```

We add *contraction* to the reduction

$$size\ (size^0\ M) = M$$

Contribution

The *environment-free* style: during computation, objects are closed by the result of their computation

- ✓ to each n-ary function f, n function inverses $f^0, f^1, \ldots\ f^n$
- ✓ the adequate reduction: $f \circ f^0 = \mathrm{id}$

Gasp's *typed evaluation*

How to evaluate an object containing function symbols?

- full evaluation, e.g. «lam ($\lambda x.\, \mathit{etaExpand}\ x$)»
- call-by-value (*contraction*), e.g. « *size* (*id* (size^0 o)) »
- weak evaluation first, e.g. « f (lam $\lambda x.\, g\ x$)»

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How to evaluate an object containing function symbols?

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- call-by-value (*contraction*), e.g. « *size* (*id* (*size*^{0} o)) »
- weak evaluation first, e.g. « f (lam $\lambda x. g\ x$)»

⇝ *"full evaluation by iterated symbolic weak evaluation"*, a.k.a *normalization-by-evaluation*
(adapted from Grégoire and Leroy [2002])

Gasp's *typed evaluation*

How to identify failures as soon as possible?

- checking the certificate *a posteriori* is not precise enough
  e.g. $\mid$ « app "m" "n" » → « z (*plus* (*size* "m") (*size* "n")) »
- errors must be detected early (during prototyping)
  # eval « *size* (lam λx. app x (app x x))» ⇝ « z (z (z o)) »

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⇝ *typed evaluation*: typing and evaluation are the same process
(dynamic typing?)

Gasp's *typed evaluation*

How to identify failures as soon as possible?

- checking the certificate *a posteriori* is not precise enough
  e.g. | « app "m" "n" » → « z (*plus* (*size* "m") (*size* "n")) »
- errors must be detected early (during prototyping)
  # `eval` « *size* (lam $\lambda x.$ app x (app x x))» ⇝ « z (z (z o)) »

⇝ *typed evaluation*: typing and evaluation are the same process
(dynamic typing?)

Contribution

✓ An evaluation algorithm `eval` performing "on-the-fly" typing of open LF objects:

  ✓ issued certificates are guaranteed to be correct
  ✓ errors are signaled *where* the certificate is ill-typed

Two case studies for Gasp

Automated proof search

An automated theorem prover based on LJT [Dyckhoff, 1992] returning NJ proof certificates:

```
# let s = «^sign ... prove : ΠA : prop. proof A = ... » ;;
# eval s « prove (imp p (imp q p)) » ;;
- : obj = « Impl p (imp q p) (λx. Impl p q λx'.x) »
# eval s « prove (imp p q) » ;;
Exception: Not_provable
```

Two case studies for Gasp

Safe type checking

A type checker for System $\mathsf{T}_{<:}$ returning a typing derivation:

```
# let s = « … infer : Πm : expr. {a : tp | is m a} = … » ;;
# eval s « infer (lam λx. s x) » ;;
- : obj = « ⟨arr nat nat, IsLam (lam λx. s x) … ⟩ »
```

Two case studies for Gasp

Safe type checking

A type checker for System $\mathsf{T}_{<:}$ returning a typing derivation:

$\#$ **let** `s` $=$ « … *infer* : $\Pi m : \mathsf{expr}.\ \{\mathsf{a} : \mathsf{tp} \mid \mathsf{is}\ m\ a\} = \ldots$ » ;;
$\#$ `eval s` « *infer* ($\mathsf{lam}\ \lambda x.\, \mathsf{s}\ x$) » ;;
- : `obj` $=$ « $\langle \mathsf{arr\ nat\ nat}, \mathsf{IsLam}\ (\mathsf{lam}\ \lambda x.\, \mathsf{s}\ x) \ldots \rangle$ »

- ✓ lightweight development (partial formalization)
- ✓ reuse off-the-shelf libraries (e.g. term indexing)

Outline

Interaction in typed program elaboration

Observations

- typed program elaboration an *interaction*

$$\text{programmer} \longleftrightarrow \text{type checker}$$

- the richer the type system is, the more expensive type checking gets (e.g. Haskell, Agda)
- typing is a *batch process* (part of compilation)
- yet, it is fed repeatedly with similar input (versions)

Interaction in typed program elaboration

Example

emacs@soupirail.inria.fr

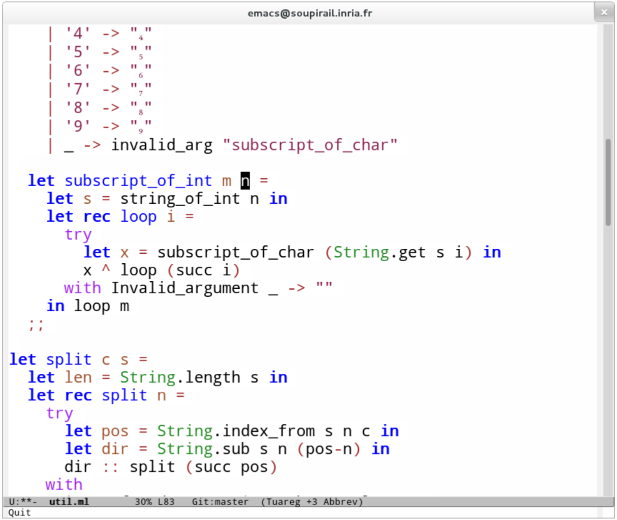
```
    | '4' -> "₄"
    | '5' -> "₅"
    | '6' -> "₆"
    | '7' -> "₇"
    | '8' -> "₈"
    | '9' -> "₉"
    | _ -> invalid_arg "subscript_of_char"

  let subscript_of_int n =
    let s = string_of_int n in
    let rec loop i =
      try
        let x = subscript_of_char (String.get s i) in
        x ^ loop (succ i)
      with Invalid_argument _ -> ""
    in loop 0
  ;;

let split c s =
  let len = String.length s in
  let rec split n =
    try
      let pos = String.index_from s n c in
      let dir = String.sub s n (pos-n) in
      dir :: split (succ pos)
    with
U:---  util.ml       30% L83   Git:master  (Tuareg +3 Abbrev)
+,-,0 for further adjustment:
```

Interaction in typed program elaboration

Example



```
emacs@soupirail.inria.fr
    | '4' -> "₄"
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    | _ -> invalid_arg "subscript_of_char"

  let subscript_of_int m n =
    let s = string_of_int n in
    let rec loop i =
      try
        let x = subscript_of_char (String.get s i) in
        x ^ loop (succ i)
      with Invalid_argument _ -> ""
    in loop m
  ;;

let split c s =
  let len = String.length s in
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    with
U:**-  util.ml      30% L83   Git:master  (Tuareg +3 Abbrev)
Quit
```

Interaction in typed program elaboration

Example

Incremental type checking

Question

How can we make type checking *incremental*?

Definition

Given a list of well-typed programs M_0, M_1,... M *and the representation of a change* δ*, decide whether* $apply(M, \delta)$ *is well-typed in less than* $|apply(M, \delta)|$.

Incremental type checking

Question

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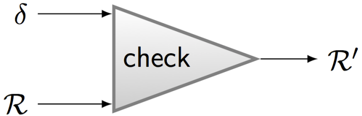
Given a list of well-typed programs $M_0, M_1, \ldots M$ *and the representation of a change* δ*, decide whether* $apply(M, \delta)$ *is well-typed in less than* $|apply(M, \delta)|$.

Hint

- save intermediate type information between runs (*context*)
- use this information in changes



Incrementality by derivation reuse

Proposition

The witness of type checking is a derivation: use it as context

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\vdash f : \mathsf{nat} \to \mathsf{nat}] \quad [\vdash x : \mathsf{nat}]}{\vdash f\,x : \mathsf{nat}}}{\vdash \lambda x. f\,x : \mathsf{nat} \to \mathsf{nat}}}{\vdash \lambda f\,x. f\,x : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \to \mathsf{nat}} \quad \dfrac{\dfrac{[\vdash x : \mathsf{nat}]}{\vdash \mathbf{s}(x) : \mathsf{nat}}}{\vdash \lambda x. \mathbf{s}(x) : \mathsf{nat} \to \mathsf{nat}}}{\vdash (\lambda f\,x. f\,x)\,(\lambda x. \mathbf{s}(x)) : \mathsf{nat}} \quad \dfrac{\dfrac{}{\vdash \mathbf{o} : \mathsf{nat}}}{\vdash \mathbf{s}(\mathbf{o}) : \mathsf{nat}}
$$

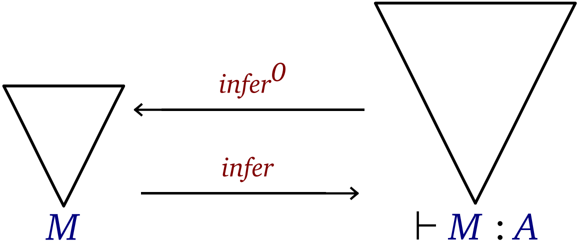
$$
\overline{\vdash (\lambda f\,x. f\,x)\,(\lambda x. \mathbf{s}(x))\,(\mathbf{s}(\mathbf{o})) : \mathsf{nat}}
$$

- it contains all intermediate type information

Incrementality by derivation reuse

Proposition

A certifying type checker in Gasp computes pieces of derivations



Incrementality by derivation reuse

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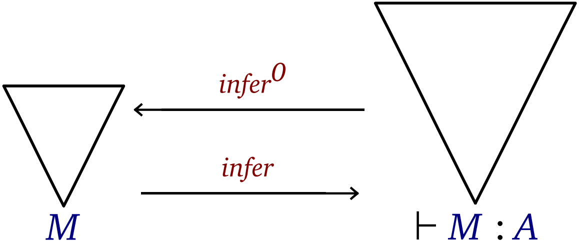
A certifying type checker in Gasp computes pieces of derivations



We need a way to

- address any *subderivation* $\mathscr{D}_i$
- reuse them in *programs* M using inverses

Naming and sharing LF objects

Contribution

- ✓ a conservative extension of LF based on *Contextual Modal Type Theory* [Nanevski et al., 2008] where objects are *sliced* in a *context* Δ of *metavariables* X
- ✓ every well-typed applicative subterm gets a metavariable *name* and can be reused by *instantiation*

Naming and sharing LF objects

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- ✓ a conservative extension of LF based on *Contextual Modal Type Theory* [Nanevski et al., 2008] where objects are *sliced* in a *context* Δ of *metavariables* X
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Example

The object $\mathsf{lam}\,(\lambda x.\,\mathsf{lam}\,(\lambda y.\,\mathsf{app}\,(x, \mathsf{app}\,(x,y))))$
is sliced into X
in the context

$$\Delta = \begin{pmatrix} X : \mathsf{tm} = \mathsf{lam}\,(\lambda x.\, Y[x/x]) \\ Y[x:\mathsf{tm}] : \mathsf{tm} = \mathsf{lam}\,(\lambda y.\, Z[x/x, y/Z[x/x, y/y]]) \\ Z[x:\mathsf{tm}, y:\mathsf{tm}] : \mathsf{tm} = \mathsf{app}\,(x,y) \end{pmatrix}$$

Example

$\#\ \ \mathit{infer}\ ((\lambda f.\, \lambda x.\, f\ x)\ (\lambda y.\, \mathsf{s}\ y)\ (\mathsf{s}\ \mathsf{o})) \rightsquigarrow \langle \mathsf{nat}, \mathscr{D} \rangle$

Example

$\#$ $\mathit{infer}\ ((\lambda f.\,\lambda x.f\ x)\ (\lambda y.\,\mathsf{s}\ y)\ (\mathsf{s}\ \mathsf{o})) \rightsquigarrow \langle \mathsf{nat}, \mathscr{D} \rangle$

$$\frac{X}{\vdash \lambda f.\,\lambda x.f\ x : \mathsf{nat}{\to}\mathsf{nat}{\to}\mathsf{nat}} \qquad \frac{Y}{\vdash \lambda y.\,\mathsf{s}\ y : \mathsf{nat}{\to}\mathsf{nat}}$$

$$[H :\ \vdash y{:}\mathsf{nat}]$$

$$\frac{Z}{\vdash \mathsf{s}\ \mathsf{o} : \mathsf{nat}} \qquad \frac{T}{\vdash \mathsf{s}\ y : \mathsf{nat}}$$

Example

$\#$ infer $((\lambda f.\, \lambda x. f\ x)\ (\lambda y.\, \mathsf{s}\ y)\ (\mathsf{s}\ \mathsf{o})) \rightsquigarrow \langle \mathsf{nat}, \mathscr{D} \rangle$

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$\#$ infer $((\lambda f.\, \lambda x. f\ x)\ (\lambda y.\, \mathsf{s}\ y)\ (\mathsf{s}\ (\mathsf{s}\ \mathsf{o})))$

Example

\# *infer* $((\lambda f.\,\lambda x.f\ x)\ (\lambda y.\,\mathsf{s}\ y)\ (\mathsf{s}\ \mathsf{o})) \rightsquigarrow \langle \mathsf{nat}, \mathscr{D} \rangle$

$$\begin{array}{c} X \\ \vdash \lambda f.\,\lambda x.f\ x : \mathsf{nat}{\to}\mathsf{nat}{\to}\mathsf{nat} \end{array} \qquad \begin{array}{c} Y \\ \vdash \lambda y.\,\mathsf{s}\ y : \mathsf{nat}{\to}\mathsf{nat} \end{array}$$

$$\begin{array}{c} Z \\ \vdash \mathsf{s}\ \mathsf{o} : \mathsf{nat} \end{array} \qquad \begin{array}{c} [H :\ \vdash y{:}\mathsf{nat}] \\ T \\ \vdash \mathsf{s}\ y : \mathsf{nat} \end{array}$$

\# *infer* $(\ X\ Y\ (\mathsf{s}\ Z))$

Example

- $\mathit{infer}\ ((\lambda f.\,\lambda x.\,f\ x)\ (\lambda y.\,\mathsf{s}\ y)\ (\mathsf{s}\ \mathsf{o})) \rightsquigarrow \langle \mathsf{nat}, \mathscr{D} \rangle$

$$\overset{X}{\vdash \lambda f.\,\lambda x.\,f\ x : \mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}} \qquad \overset{Y}{\vdash \lambda y.\,\mathsf{s}\ y : \mathsf{nat}\to\mathsf{nat}}$$

$$\overset{Z}{\vdash \mathsf{s}\ \mathsf{o} : \mathsf{nat}} \qquad \begin{matrix}[H : \ \vdash y{:}\mathsf{nat}] \\ \overset{T}{\vdash \mathsf{s}\ y : \mathsf{nat}}\end{matrix}$$

- $\mathit{infer}\ ((\mathit{infer}^0\ X)\ (\mathit{infer}^0\ Y)\ (\mathsf{s}\ (\mathit{infer}^0\ Z)))$

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Example

$\mathit{infer}\ ((\lambda f.\,\lambda x.\,f\ x)\ (\lambda y.\,\mathsf{s}\ y)\ (\mathsf{s}\ \mathsf{o})) \rightsquigarrow \langle \mathsf{nat}, \mathscr{D} \rangle$

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Example

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$\#\ \mathit{infer}\ ((\mathit{infer}^0\ X)\ (\lambda y.\,\mathsf{s}\ (\mathit{infer}^0\ T[H/\mathit{infer}\ y]))\ (\mathit{infer}^0\ Z))$

Contributions

- ✓ Gasp: certifying type checker ⟶ incremental type checking
- ✓ sharing computation results by *function inverses*
- ✓ a safe approach: (shared) type derivation always available

Outline

Contributions

- ✓ Gasp, a library for manipulating LF proof certificates
- ✓ support for *environment-free* style thanks to *function inverses*
- ✓ its extension to proof reuse enabling *incremental type checking*

Contributions

- ✓ Gasp, a library for manipulating LF proof certificates
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Other contributions:

- ✓ *inter-deriving* sequent calculi from natural deduction, using off-the-shelf *program transformations*:

$$\text{type checker for N.D.} \xrightarrow{\text{compilation}} \text{type checker for S.C.}$$

- ✓ an original metatheory of *spine-form* LF

Perspectives

- isolate higher-order term manipulation library
  put the *locally named* pattern into practice
- investigate typing of inverse functions
  and their relation with *NbE*
- front-end editor generating *deltas* ("*structured editor*")
  safe refactoring tools, typed version control
- LCF-style interactive theorem prover based on LF
  tactics as OCaml functions

Thank you

Backup slides

Certifying software

a.k.a *Proof-Carrying Code* [Necula, 1997]

certified
: program together with proof that it respects the specification on all input (Coq, Matita...)

certifying
: black box, emits a proof certificate verifiable a posteriori, but not guaranteed to be correct

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: black box, emits a proof certificate verifiable a posteriori, but not guaranteed to be correct

Advantages of the certifying scheme

- same safety (but different quality of implementation)
- program source need not be revealed
- more lightweight (partial formalization)
  e.g. no verification of graph coloring, term indexing…

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